Canonical quantization of the covariant fields: the Dirac field on the de Sitter spacetime

Ion I. Cotaescu

Abstract

The properties of the covariant fields on the de Sitter spacetimes are investigated focusing on the isometry generators and Casimir operators in order to establish the equivalence among the covariant representations (CR) and the unitary irreducuble ones (UIR) of the de Sitter isometry group. For the Dirac field it is shown that the spinor CR transforming the Dirac field under de Sitter isometries is equivalent to a direct sum of two UIRs of the $Sp(2,2)$ group transforming alike the particle and antiparticle field operators in momentum representation (rep.). Their basis generators and Casimir operators are written down finding that these reps. are equivalent to a UIR from the principal series whose canonical labels are determined by the fermion mass and spin.

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Contents

1. Introduction

Our world is governed by four fundamental interactions that can be investigated at two levels:
- ***the classical level*** of ***continuous matter*** neutral or electrically charged,
- ***the quantum level*** of interacting ***quantum fields*** whose quanta are the ***elementary particles***, fermions of spin 1/2 and gauge bosons of spin 1.

The actual physical evidences (up to the scale of the LHC energies $\sim 10^{-20}m$) give the following picture

CLASSICAL	QUANTUM
Gravity	NO DATA
Electromagnetism	Electromagnetism
NO DATA	Nuclear weak
NO DATA	Nuclear strong

At this scale we can adopt ***the semi-classical picture*** of ***quantum fields*** in the presence of the classical gravity of ***curved*** spacetimes without torsion.

The principal measured quantities are the conserved ones corresponding to symmetries (via Noether theorem). These are the ***mass***, the ***spin*** and different ***charges***.

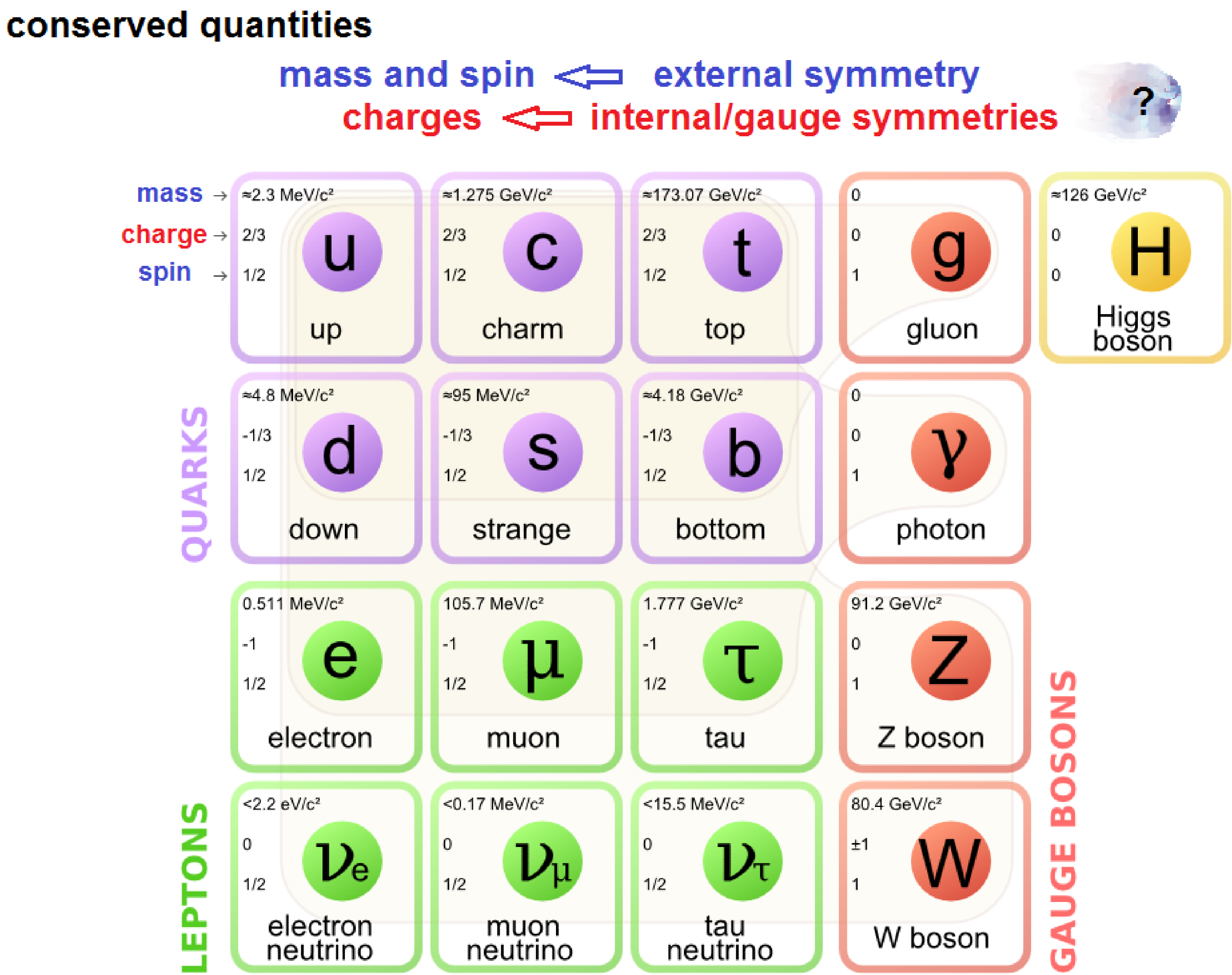

Figure 1: Symmetries determining conserved quantities

It is known that in special relativity the isometries play a crucial role in quantizing free fields since the principal particle properties, the mass and spin, are eigenvalues of the Casimir operators of the Poincaré group. Then it is natural to ask what happens in the case of the curved spacetimes - ***how the mass and spin can be defined by the isometry invariants***. The answer could be found taking into account that:

1. The theory of ***quantum fields with spin*** on curved $(1+3)$-dimensional local-Minkowskian manifolds (M,g), having the Minkowski flat model (M_0,η), can be correctly constructed only in ***orthogonal*** (non-holonomic) ***local frames*** [1, 2].

2. The entire theory must be invariant under the ***orthogonal*** transformations of the local frames, i. e. ***gauge*** transformations that form the ***gauge group*** $G(\eta)=SO(1,3)$ which is the isometry group of the flat model M_0.

3. Since the transformations of the ***isometry group*** $I(M)$ can change this gauge we proposed to enlarge the concept of isometry considering ***external symmetry*** transformations that preserve not only the metric but the gauge too [3, 4]. The group of external symmetry $S(M)$ is isomorphic to the ***universal covering group*** of the isometry group $I(M)$.

4. The quantum fields transform according to the ***covariant reps.*** (CRs) of the group $S(M)$ which are ***induced*** by the (non-unitary) finite-dimensional reps. of the group $\mathrm{Spin}(1,3) \sim SL(2,\mathbb{C})$, i. e. the ***universal covering group*** of the gauge group $G(\eta) = L_+^\uparrow \subset SO(1,3)$ [3]-[5].

5. The conserved observables are the generators of the CRs, i. e. the differential operators produced by the ***Killing vectors*** (associated to isometries) according to the ***generalized*** Carter and McLenagan formula [25, 3, 4].

The main purpose of the present talk is to present the general theory of the CRs pointing out the role of their generators in canonical quantization.

The examples we give are the well-known case of ***special relativity*** as well as the CRs on the $(1+3)$-dimensional ***de Sitter spacetime*** where the specific $SO(1,4)$ isometries generate conserved observables with a well-defined physical meaning [20] that allow us to perform the canonical quantization just as in special relativity.

We must stress that in the case of the de Sitter spacetime our theory of induced CRs, we use here, is equivalent to that proposed by Nachtmann [16] many years ago, and is completely different from other approaches [13]-[15] that are using ***linear reps.*** of the universal covering group of the de Sitter isometry group [9, 10].

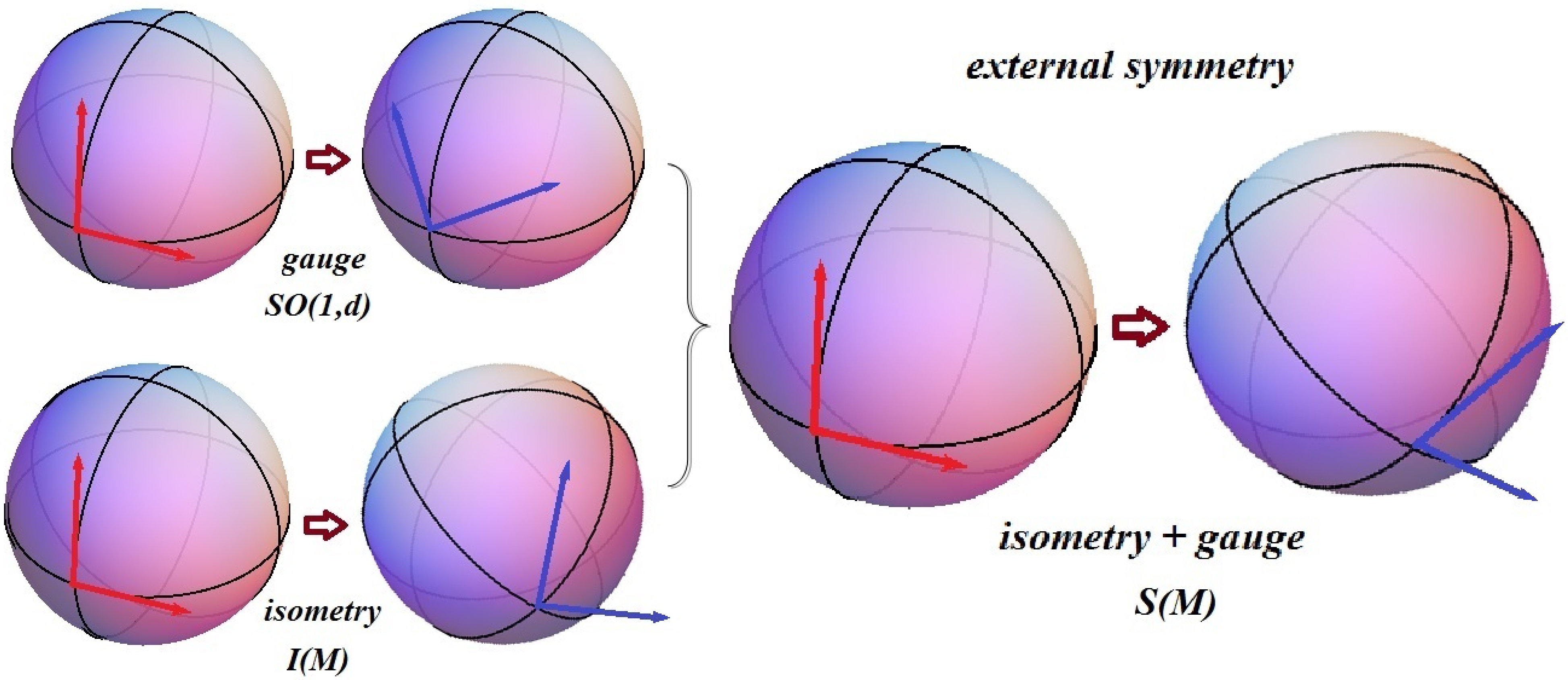

Figure 2: The ***external symmetry*** combines the ***isometries*** with suitable ***gauge*** transformations preserving thus the metric and the ***relative positions*** of the local frames with respect to the natural ones.

2. Covariant quantum fields

Natural and local frames

Let (M, g) be a $(1+3)$-dimensional local-Minkowskian spacetime equipped with ***local*** frames $\{x; e\}$ formed by a ***local chart*** (or natural frame) $\{x\}$ and a ***non-holonomic orthogonal frame*** $\{e\}$.

The coordinates x^μ of the local chart are labelled by ***natural*** indices $\mu, \nu, ... = 0, 1, 2, 3$. The orthogonal frames are defined by the vector fields,

$$e_{\hat{\alpha}} = e^\mu_{\hat{\alpha}} \partial_\mu \,, \tag{1}$$

while the corresponding coframes are defined by the 1-forms

$$\omega^{\hat{\alpha}} = \hat{e}^{\hat{\alpha}}_\mu dx^\mu \,. \tag{2}$$

These are called ***tetrad*** fields, are labelled by ***local*** indices, $\hat{\alpha}, ...\hat{\mu}, \hat{\nu}, ... = 0, 1, 2, 3$ and obey the usual ***duality*** relations

$$\hat{e}^{\hat{\mu}}_\alpha \, e^\alpha_{\hat{\nu}} = \delta^{\hat{\mu}}_{\hat{\nu}} \,, \quad \hat{e}^{\hat{\mu}}_\alpha \, e^\beta_{\hat{\mu}} = \delta^\beta_\alpha \,, \tag{3}$$

and the ***orthonormalization*** conditions,

$$e_{\hat{\mu}} \cdot e_{\hat{\nu}} \stackrel{def}{=} g_{\alpha\beta} e^{\alpha}_{\hat{\mu}} e^{\beta}_{\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}} \, , \quad \omega^{\hat{\mu}} \cdot \omega^{\hat{\nu}} \stackrel{def}{=} g^{\alpha\beta} \hat{e}^{\hat{\mu}}_{\alpha} \hat{e}^{\hat{\nu}}_{\beta} = \eta^{\hat{\mu}\hat{\nu}} \, , \tag{4}$$

where $\eta = \mathrm{diag}(1, -1, -1, -1)$ is the metric of the Minkowski model (M_0, η) of (M, g). Then the line element can be written as

$$ds^2 = \eta_{\hat{\alpha}\hat{\beta}} \omega^{\hat{\alpha}} \omega^{\hat{\beta}} = g_{\mu\nu} dx^{\mu} dx^{\nu} \, . \tag{5}$$

which means that $g_{\mu\nu} = \eta_{\hat{\alpha}\hat{\beta}} \hat{e}^{\hat{\alpha}}_{\mu} \hat{e}^{\hat{\beta}}_{\nu}$. This is the metric tensor which raises or lowers the natural indices while for the local ones we have to use the flat metric η.

The vector fields $e_{\hat{\nu}}$ satisfy the commutation rules

$$[e_{\hat{\mu}}, e_{\hat{\nu}}] = e^{\alpha}_{\hat{\mu}} e^{\beta}_{\hat{\nu}} (\hat{e}^{\hat{\sigma}}_{\alpha,\beta} - \hat{e}^{\hat{\sigma}}_{\beta,\alpha}) \hat{\partial}_{\hat{\sigma}} = C_{\hat{\mu}\hat{\nu}\cdot}^{\cdot\cdot\hat{\sigma}} \hat{\partial}_{\hat{\sigma}} \tag{6}$$

defining the Cartan coefficients which help us to write the ***conecttion coefficients in local frames*** as

$$\hat{\Gamma}^{\hat{\sigma}}_{\hat{\mu}\hat{\nu}} = e^{\alpha}_{\hat{\mu}} e^{\beta}_{\hat{\nu}} (\hat{e}^{\hat{\sigma}}_{\gamma} \Gamma^{\gamma}_{\alpha\beta} - \hat{e}^{\hat{\sigma}}_{\beta,\alpha}) = \frac{1}{2} \eta^{\hat{\sigma}\hat{\lambda}} (C_{\hat{\mu}\hat{\nu}\hat{\lambda}} + C_{\hat{\lambda}\hat{\mu}\hat{\nu}} + C_{\hat{\lambda}\hat{\nu}\hat{\mu}}) \, . \tag{7}$$

We specify that this connection is often called spin connection (and denoted by $\Omega^{\hat{\sigma}}_{\hat{\mu}\hat{\nu}}$) but it is the same as the natural one. The notation Γ stands for the usual Christoffel symbols representing the ***connection coefficients in natural frames***.

Covariant fields on curved spacetimes

The metric η remains invariant under the transformations of the group $O(1,3)$ which includes as subgroup the ***gauge group*** $G(\eta)=L_+^\uparrow$, whose universal covering group is $\mathrm{Spin}(1,3) = SL(2,\mathbb{C})$. In the usual covariant parametrization, with the real parameters, $\tilde\omega^{\hat\alpha\hat\beta}=-\tilde\omega^{\hat\beta\hat\alpha}$, the transformations

$$A(\tilde\omega)=\exp\left(-\frac{i}{2}\,\tilde\omega^{\hat\alpha\hat\beta}S_{\hat\alpha\hat\beta}\right)\in SL(2,\mathbb{C}) \tag{8}$$

depend on the covariant basis-generators $S_{\hat\alpha\hat\beta}$ of the $sl(2,\mathbb{C})$ Lie algebra which satisfy

$$[S_{\hat\mu\hat\nu},\, S_{\hat\sigma\hat\tau}]=i(\eta_{\hat\mu\hat\tau}\,S_{\hat\nu\hat\sigma}-\eta_{\hat\mu\hat\sigma}\,S_{\hat\nu\hat\tau}+\eta_{\hat\nu\hat\sigma}\,S_{\hat\mu\hat\tau}-\eta_{\hat\nu\hat\tau}\,S_{\hat\mu\hat\sigma})\,. \tag{9}$$

The matrix elements in local frames of the $SO(1,3)$ transformation associated to $A(\tilde\omega)$ through the ***canonical*** homomorphism can be expanded as

$$\Lambda^{\hat\mu\,\cdot}_{\cdot\,\hat\nu}(\tilde\omega)=\delta^{\hat\mu}_{\hat\nu}+\tilde\omega^{\hat\mu\,\cdot}_{\cdot\,\hat\nu}+\cdots\in SO(1,3) \tag{10}$$

We denote by $I=A(0)\in SL(2,\mathbb{C})$ and $1=\Lambda(0)\in SO(1,3)$ the identity transformations of these groups.

The ***covariant fields***, $\psi_{(\rho)} : M \to \mathcal{V}_{(\rho)}$, are locally defined over M with values in the vector spaces $\mathcal{V}_{(\rho)}$ carrying the ***finite-dimensional non-unitary reps.*** ρ of the group $SL(2,\mathbb{C})$ (briefly presented in the Appendix A). In general, these representations are reducible as arbitray sums of irreducible ones, (j_1, j_2). For example, the vector field transforms according to the irreducible rep. $\rho_v = (1/2, 1/2)$ while for the Dirac field we use the reducible rep. $\rho_s = (1/2, 0) \oplus (0, 1/2)$.

The covariant derivatives of the field $\psi_{(\rho)}$ in local frames (or natural ones),

$$D_{\hat\alpha}^{(\rho)} = e_{\hat\alpha}^{\mu} D_{\mu}^{(\rho)} = e_{\hat\alpha}^{\mu}\partial_{\mu} + \frac{i}{2}\rho(S^{\hat\beta\,\cdot}_{\cdot\,\hat\gamma})\,\hat\Gamma^{\hat\gamma}_{\hat\alpha\hat\beta}\,, \tag{11}$$

assure the ***covariance*** of the whole theory under the (point-dependent) ***gauge transformations***,

$$\omega(x) \to \Lambda[A(x)]\omega(x) \tag{12}$$

$$\psi_{(\rho)}(x) \to \rho[A(x)]\psi_{(\rho)}(x), \tag{13}$$

produced by the ***sections*** $A : M \to SL(2,\mathbb{C})$ of the spin fiber bundle.

Note that in the case of the vector and tensor fields the local derivatives coincide with the covariant ones acting as $D_\mu T^{\cdots\nu\cdots} = \partial_\mu T^{\cdots\nu\cdots}\cdots + \Gamma^{\nu}_{\mu\alpha}T^{\cdots\alpha\cdots} + \cdots$. This means that the local frames are needful only in the case of the field with half integer spin.

(M,g) may have ***isometries***, $x \to x' = \phi_{\mathfrak{g}}(x)$, given by the (non-linear) rep. $\mathfrak{g} \to \phi_{\mathfrak{g}}$ of the isometry group $I(M)$ with the composition rule $\phi_{\mathfrak{g}} \circ \phi_{\mathfrak{g}'} = \phi_{\mathfrak{g}\mathfrak{g}'}$, $\forall \mathfrak{g}, \mathfrak{g}' \in I(M)$. Then we denote by $id = \phi_{\mathfrak{e}}$ the identity function, corresponding to the unit $\mathfrak{e} \in I(M)$, and deduce $\phi_{\mathfrak{g}}^{-1} = \phi_{\mathfrak{g}^{-1}}$. In a given ***parametrization***, $\mathfrak{g} = \mathfrak{g}(\xi)$ (with $\mathfrak{e} = \mathfrak{g}(0)$), the isometries

$$x \to x' = \phi_{\mathfrak{g}(\xi)}(x) = x + \xi^a k_a(x) + \ldots \qquad (14)$$

lay out the ***Killing vectors*** $k_a = \partial_{\xi_a} \phi_{\mathfrak{g}(\xi)}|_{\xi=0}$ associated to the parameters ξ^a $(a, b, \ldots = 1, 2 \ldots N)$.

The isometries may change the relative position of the local frames. For this reason we proposed the theory of external symmetry [3] where the ***combined transformations*** $(A_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ are able to correct the position of the local frames ***preserving thus not only the metric but the gauge too***, i. e.

$$\Rightarrow \quad \omega(x) \to \omega'(x') \stackrel{def}{=} \omega[\phi_{\mathfrak{g}}(x)] = \Lambda[A_{\mathfrak{g}}(x)]\omega(x)\,. \qquad (15)$$

Hereby, we deduce [3],

$$\Lambda^{\hat{\alpha}\,\cdot}_{\cdot\,\hat{\beta}}[A_{\mathfrak{g}}(x)] = \hat{e}^{\hat{\alpha}}_{\mu}[\phi_{\mathfrak{g}}(x)] \frac{\partial \phi^{\mu}_{\mathfrak{g}}(x)}{\partial x^{\nu}} \, e^{\nu}_{\hat{\beta}}(x)\,, \qquad (16)$$

assuming, in addition, that $A_{\mathfrak{g}=\mathfrak{e}}(x)=1$. We obtain thus the desired ***transformation laws under isometries***,

$$(A_{\mathfrak{g}},\phi_{\mathfrak{g}}): \qquad \begin{array}{rcl} e(x) \to & e'(x') &= e[\phi_{\mathfrak{g}}(x)]\,, \\ \psi_{(\rho)}(x) \to & \psi'_{(\rho)}(x') &= \rho[A_{\mathfrak{g}}(x)]\psi_{(\rho)}(x)\,. \end{array} \tag{17}$$

that ***preserve the gauge***.

The set of combined transformations $(A_{\mathfrak{g}},\phi_{\mathfrak{g}})$ form the ***group of external symmetry***, denoted by $S(M)$. This is isomorphic with the universal covering group of $I(M)$. The multiplication rule is defined as

$$(A_{\mathfrak{g}'},\phi_{\mathfrak{g}'})*(A_{\mathfrak{g}},\phi_{\mathfrak{g}}) \stackrel{def}{=} ((A_{\mathfrak{g}'}\circ\phi_{\mathfrak{g}})\times A_{\mathfrak{g}},\phi_{\mathfrak{g}'}\circ\phi_{\mathfrak{g}}) = (A_{\mathfrak{g}'\mathfrak{g}},\phi_{\mathfrak{g}'\mathfrak{g}})\,, \tag{18}$$

such that the unit element is $(A_{\mathfrak{e}},\phi_{\mathfrak{e}}) = (I,id)$ while the inverse of the element $(A_{\mathfrak{g}},\phi_{\mathfrak{g}})$ reads $(A_{\mathfrak{g}},\phi_{\mathfrak{g}})^{-1} = (A_{\mathfrak{g}^{-1}}\circ\phi_{\mathfrak{g}^{-1}},\phi_{\mathfrak{g}^{-1}})$. For other mathematical details see Ref. [3].

In a given parametrization, $\mathfrak{g}=\mathfrak{g}(\xi)$, for small values of ξ^a, the $SL(2,\mathbb{C})$ parameters of $A_{\mathfrak{g}(\xi)}(x)\equiv A[\tilde\omega_\xi(x)]$ can be expanded as $\tilde\omega_\xi^{\hat\alpha\hat\beta}(x)=\xi^a\Omega_a^{\hat\alpha\hat\beta}(x)+\cdots$ where

$$\Omega_a^{\hat\alpha\hat\beta}\equiv \frac{\partial\tilde\omega_\xi^{\hat\alpha\hat\beta}}{\partial\xi^a}_{|\xi=0} = \left(\hat e^{\hat\alpha}_\mu\, k^\mu_{a,\nu}+\hat e^{\hat\alpha}_{\nu,\mu}k^\mu_a\right) e^\nu_{\hat\lambda}\eta^{\hat\lambda\hat\beta} \tag{19}$$

are skew-symmetric functions, $\Omega_a^{\hat{\alpha}\hat{\beta}} = -\Omega_a^{\hat{\beta}\hat{\alpha}}$, only when k_a are Killing vectors [3].

The last of Eqs. (17) defines the CRs ***induced*** by the finite-dimensional rep., ρ, of the group $SL(2,\mathbb{C})$. These are operator-valued reps., $T^{(\rho)} : (A_{\mathfrak{g}}, \phi_{\mathfrak{g}}) \to T_{\mathfrak{g}}^{(\rho)}$, of the group $S(M)$ whose ***covariant transformations***,

$$\Rightarrow \quad (T_{\mathfrak{g}}^{(\rho)}\psi_{(\rho)})[\phi_{\mathfrak{g}}(x)] = \rho[A_{\mathfrak{g}}(x)]\psi_{(\rho)}(x)\,, \tag{20}$$

leave the ***field equation invariant*** since their basis-generators [3],

$$\Rightarrow \quad X_a^{(\rho)} = i\partial_{\xi^a} T^{(\rho)}_{\mathfrak{g}(\xi)|\xi=0} = -ik_a^\mu \partial_\mu + \frac{1}{2}\Omega_a^{\hat{\alpha}\hat{\beta}}\rho(S_{\hat{\alpha}\hat{\beta}})\,, \tag{21}$$

commute with the operator of the field equation. These satisfy the commutation rules

$$[X_a^{(\rho)}, X_b^{(\rho)}] = ic_{abc}X_c^{(\rho)} \tag{22}$$

where c_{abc} are the structure constants of the algebras $s(M) \sim i(M)$ - they are the basis-generators of a CR of the $s(M)$ algebra ***induced*** by the rep. ρ of the $\mathrm{spin}(1,3) = sl(2,\mathbb{C})$ algebra.

These generators can be put in (general relativistic) ***covariant*** form either in non-holonomic frames [3],

$$\Rightarrow \quad X_a^{(\rho)} = -ik_a^\mu D_\mu^{(\rho)} + \frac{1}{2} k_{a\,\mu;\nu}\, e_{\hat{\alpha}}^\mu\, e_{\hat{\beta}}^\nu\, \rho(S^{\hat{\alpha}\hat{\beta}})\,, \tag{23}$$

or even in holonomic ones [4], generalizing thus the formula given by Carter and McLenaghan for the Dirac field [25].

The generators (21) have, in general, point-dependent spin terms which do not commute with the orbital parts. However, there are tetrad-gauges in which at least the generators of a subgroup $H \subset I(M)$ may have ***point-independent spin terms*** commuting with the orbital parts. Then we say that the restriction to H of the CR $T^{(\rho)}$ is ***manifest covariant*** [3].

Obviously, if $H = I(M)$ then the whole rep. $T^{(\rho)}$ is manifest covariant. In particular, the linear CRs on the Minkowski spacetime have this property. This gives rise to the so called ***Lorentz covariance*** which, according to our theory, is ***universal*** for any $(1+3)$-dimensional local-Minkowskian manifold.

Lagrangian formalism

In the Lagrangian theory the Lagrangian densities must be invariant positively defined quantities. Since the finite-dimensional reps. ρ of the $SL(2,\mathbb{C})$ group are non-unitary, we must use the (generalized) ***Dirac conjugation***, $\overline{\psi}_{(\rho)} = \psi_\rho^+ \gamma_{(\rho)}$, where the matrix $\gamma_{(\rho)} = \gamma_{(\rho)}^+ = \gamma_{(\rho)}^{-1}$ satisfies $\overline{\rho}(A) = \gamma_{(\rho)}\rho(A)^+\gamma_{(\rho)} = \rho(A^{-1})$. Then the form $\overline{\psi}_{(\rho)}\psi_{(\rho)}$ is invariant under the gauge transformations. In general, the Dirac conjugation can be defined for the reducible reps. of the form $\rho = ...(j_1, j_2) \oplus (j_2, j_1)...$.

The covariant equations of the free fields can be derived from actions of the form

$$\mathcal{S}[\psi_{(\rho)}, \overline{\psi}_{(\rho)}] = \int_\Delta d^4x \sqrt{g}\, \mathcal{L}(\psi_{(\rho)}, \psi_{(\rho);\mu}, \overline{\psi}_{(\rho)}, \overline{\psi}_{(\rho);\mu})\,, \quad g = |\det g_{\mu\nu}|\,, \quad (24)$$

depending on the field $\psi_{(\rho)}$, its Dirac adjoint $\overline{\psi}_{(\rho)}$ and their corresponding covariant derivatives $\psi_{(\rho);\mu} = D_\mu^{(\rho)}\psi_{(\rho)}$ and $\overline{\psi}_{(\rho);\mu} = \overline{D_\mu^{(\rho)}\psi_{(\rho)}}$ defined by the rep. ρ of the group $SL(2,\mathbb{C})$.

The action $\mathcal{S}$ is extremal if the covariant fields satisfy the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}_{(\rho)}} - \frac{1}{\sqrt{g}}\,\partial_\mu \frac{\partial(\sqrt{g}\,\mathcal{L})}{\partial \overline{\psi}_{(\rho),\mu}} = 0\,, \qquad \frac{\partial \mathcal{L}}{\partial \psi_{(\rho)}} - \frac{1}{\sqrt{g}}\,\partial_\mu \frac{\partial(\sqrt{g}\,\mathcal{L})}{\partial \psi_{(\rho),\mu}} = 0\,. \tag{25}$$

Any transformation $\psi_{(\rho)} \to \psi'_{(\rho)} = \psi_{(\rho)} + \delta\psi_{(\rho)}$ leaving the action invariant, $\mathcal{S}[\psi'_{(\rho)}, \overline{\psi}'_{(\rho)}] = \mathcal{S}[\psi_{(\rho)}, \overline{\psi}_{(\rho)}]$, is a ***symmetry*** transformation. The Noether theorem shows that each symmetry transformation gives rise to the ***current***

$$\Theta^\mu \;\propto\; \delta\overline{\psi}_{(\rho)} \frac{\partial \mathcal{L}}{\partial \overline{\psi}_{(\rho),\mu}} + \frac{\partial \mathcal{L}}{\partial \psi_{(\rho),\mu}}\,\delta\psi_{(\rho)} \tag{26}$$

which is ***conserved*** in the sense that ${\Theta^\mu}_{;\mu} = 0$.

In the case of isometries we have $\delta\psi_{(\rho)} = -i\xi^a X_a^{(\rho)}\psi_{(\rho)}$. Consequently, each isometry of parameter ξ^a give rise to the corresponding conserved current

$$\Theta^\mu_a = i\left(\overline{X_a^{(\rho)}\psi_{(\rho)}}\,\frac{\partial \mathcal{L}}{\partial \overline{\psi}_{(\rho),\mu}} - \frac{\partial \mathcal{L}}{\partial \psi_{(\rho),\mu}}\,X_a^{(\rho)}\psi_{(\rho)}\right)\,, \quad a = 1, 2...N\,. \tag{27}$$

Then we may define the ***relativistic scalar product*** $\langle\ ,\ \rangle$ as

$$\Rightarrow \quad \langle \psi, \psi' \rangle = i \int_{\partial\Delta} d\sigma_\mu \sqrt{g} \left(\overline{\psi} \frac{\partial \mathcal{L}'}{\partial \overline{\psi'}_{,\mu}} - \frac{\partial \mathcal{L}}{\partial \psi_{,\mu}} \psi' \right) , \tag{28}$$

such that the conserved quantities (charges) can be represented as ***expectation values*** of ***isometry generators***,

$$\Rightarrow \quad C_a = \int_{\partial\Delta} d\sigma_\mu \sqrt{g}\, \Theta^\mu_a = \langle \psi_{(\rho)}, X^{(\rho)}_a \psi_{(\rho)} \rangle \, , \tag{29}$$

Notice that the operators X are ***self-adjoint*** with respect to this scalar product, i. e. $\langle X\psi, \psi' \rangle = \langle \psi, X\psi' \rangle$.

From the algebra freely generated by the isometry generators we may select the sets of commuting operators $\{A_1, A_2, ...A_n\}$ determining the ***fundamental solutions*** of ***particles***, $U_\alpha \in \mathcal{F}^+$, and ***antiparticles***, $V_\alpha \in \mathcal{F}^-$, that depend on the set of the corresponding eigenvalues $\alpha = \{a_1, a_2, ...a_n\}$ spanning a discrete or continuous spectra of the common eigenvalue problems

$$A_i U_\alpha = a_i U_\alpha \, , \quad A_i V_\alpha = -a_i V_\alpha \, , \quad i = 1, 2...n \, . \tag{30}$$

After determining the fundamental solutions we may write the ***mode expansion***

$$\Rightarrow \quad \psi_{(\rho)}(x) = \left(\sum_{\alpha \in \Sigma_d} + \int_{\alpha \in \Sigma_c} d(\alpha) \right) \left[U_\alpha(x) a(\alpha) + V_\alpha(x) b^*(\alpha) \right] , \tag{31}$$

where we sum over the discrete part (Σ_d) and integrate over the continuous part (Σ_c) of the spectrum $\Sigma = \Sigma_d \cup \Sigma_c$.

The fundamental solutions are orthogonal with respect to the relativistic scalar product and can be normalized such that

$$\langle U_\alpha, U_{\alpha'} \rangle = \pm \langle V_\alpha, V_{\alpha'} \rangle = \delta(\alpha, \alpha') = \begin{cases} \delta_{\alpha,\alpha'} & \text{if } \alpha, \alpha' \in \Sigma_d \\ \delta(\alpha - \alpha') & \text{if } \alpha, \alpha' \in \Sigma_c \end{cases} \tag{32}$$

$$\langle U_\alpha, V_{\alpha'} \rangle = \langle V_\alpha, U_{\alpha'} \rangle = 0 \, , \tag{33}$$

where the sign $+$ arises for fermions while the sign $-$ is obtained for bosons.

Canonical quantization

The theory get a physical meaning only after performing the ***second quantization*** postulating canonical non-vanishing rules (with the notation $[x,y]_\pm = xy \pm yx$) as

$$\Big[a(\alpha), a^\dagger(\alpha')\Big]_\pm = \Big[b(\alpha), b^\dagger(\alpha')\Big]_\pm = \delta(\alpha,\alpha')\,. \tag{34}$$

Then the fields $\psi_{(\rho)}$ become quantum fields (with $b^\dagger$ instead of b^*) while the conserved quantities (29) become one-particle operators,

$$\Rightarrow \quad C_a \to \mathbf{X}_a^{(\rho)} =: \langle \psi_{(\rho)}, X_a^{(\rho)} \psi_{(\rho)} \rangle : \tag{35}$$

calculated respecting the normal ordering of the operator products [26]. Now the one particle operators $\mathbf{X}_a^{(\rho)}$ are the basis generators of a rep. of the algebra $s(M)$ with values in operator algebra. In a similar manner one can define the generators of the ***internal symmetries*** as for example the charge one-particle operator $\mathbf{Q} =: \langle \psi, \psi \rangle$:.

Thus we obtain a reach operator algebra formed by field operators and the one-particle ones which have the obvious properties

$$[\mathbf{X}, \psi(x)] = -(X\psi)(x)\,, \quad [\mathbf{X}, \mathbf{Y}] =: \langle \psi, ([X,Y]\psi) \rangle : \;. \tag{36}$$

In general, if the one-particle operator X does not mix among themselves the subspaces of fundamental solutions it can be expanded as

$$\begin{aligned}\mathbf{X} &= :\langle\psi, X\psi\rangle := \mathbf{X}^{(+)} + \mathbf{X}^{(-)} \\ &= \int_{\alpha\in\Sigma}\int_{\alpha'\in\Sigma} \tilde{X}^{(+)}(\alpha,\alpha')a^\dagger(\alpha)a(\alpha') + \tilde{X}^{(-)}(\alpha,\alpha')b^\dagger(\alpha)b(\alpha')\,,\end{aligned} \tag{37}$$

where

$$\tilde{X}^{(+)}(\alpha,\alpha') = \langle U_\alpha, XU_{\alpha'}\rangle\,,\quad \tilde{X}^{(-)}(\alpha,\alpha') = \langle V_\alpha, XV_{\alpha'}\rangle\,. \tag{38}$$

When there are differential operators $\tilde{X}^{(\pm)}$ acting on the continuous variables of the set $\{\alpha\}$ such that $\tilde{X}^{(\pm)}(\alpha,\alpha') = \delta(\alpha,\alpha')\tilde{X}^{(\pm)}$ we say that $\tilde{X}^{(\pm)}$ are the operators of the ***rep.*** $\{\alpha\}$ (in the sense of the relativistic QM).

We stress that all the isometry generators have this property such that the corresponding operators $\tilde{X}_a^{(\pm)}$ are the basis generators of the isometry transformations of the field operators a and b . However, the algebraic relations (34) remain invariant only if a and b transform according to UIRs of the isometry group.

A crucial problem is now the equivalence between the CR transformimg the covariant field $\psi_{(\rho)}$ and the set of UIRs transforming the particle and antiparticle operators a and b. This will be referred here as the ***CR-UIR equivalence***.

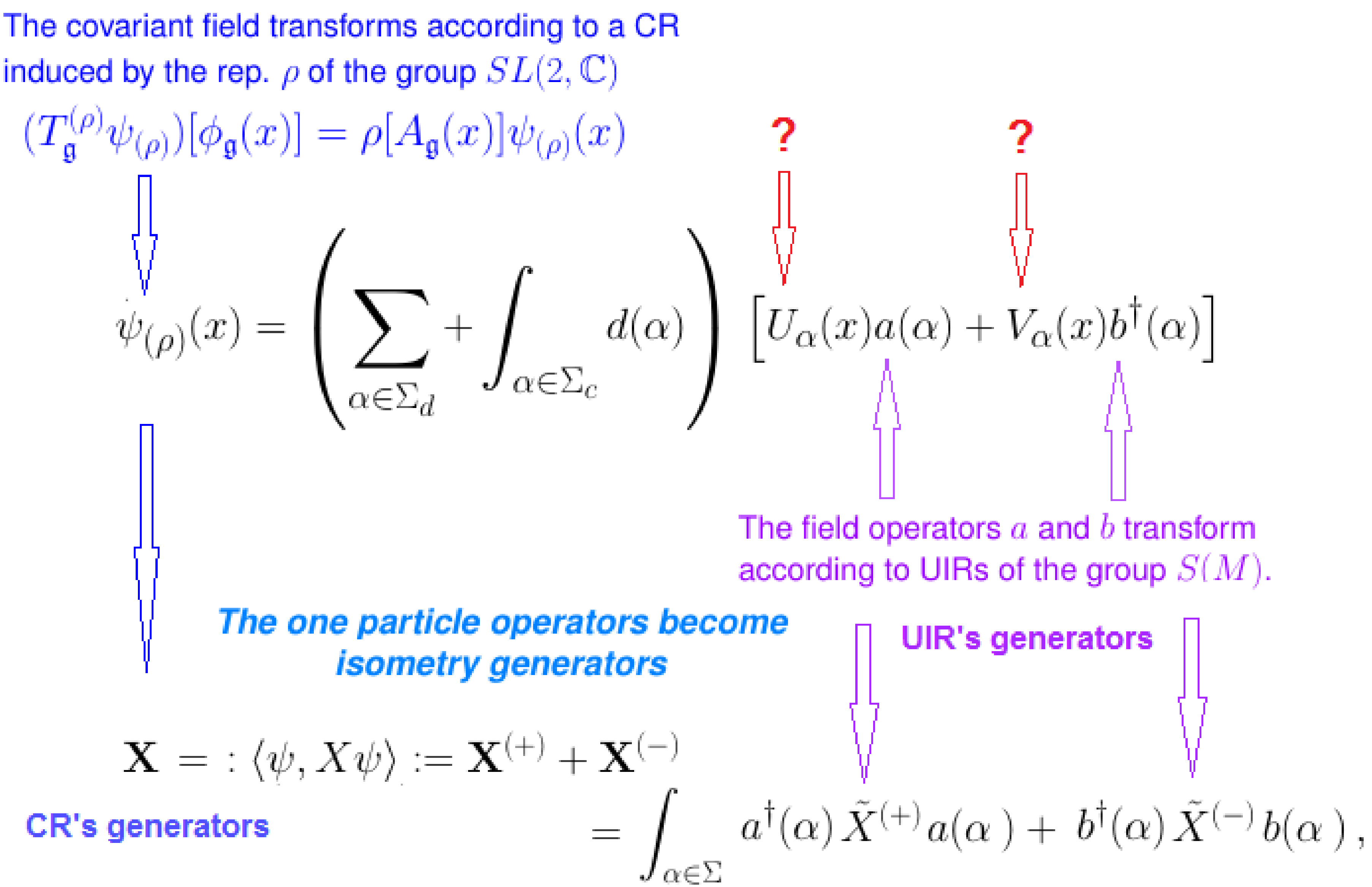

Figure 3: The CR-UIR equivalence of the reps. of the group $S(M)$ and its algebra $s(M)$

3. Covariant fields in special relativity

The problem of CR-UIR equivalence is successfully solved in special relativity thanks to the Wigner theory of induced reps. of the Poincaré group.

On the Minkowski spacetime, (M_0, η), the fields $\psi_{(\rho)}$ transform under isometries according to ***manifest*** CRs in ***inertial*** (local) frames defined by $e^\mu_\nu = \hat{e}^\mu_\nu = \delta^\mu_\nu$.

Generators of manifest CRs

The isometries are just the transformations $x \to x' = \Lambda[A(\omega)]x - a$ of the Poincaré group $I(M_0) = \mathcal{P}^\uparrow_+ = T(4) \circledS L^\uparrow_+$ [24] whose universal covering group is $S(M_0) = \tilde{\mathcal{P}}^\uparrow_+ = T(4) \circledS SL(2,\mathbb{C})$. The manifest CRs, $T^{(\rho)} : (A,a) \to T^{(\rho)}_{A,a}$, of the $S(M_0)$ group have the transformation rules

$$\Rightarrow \quad (T^{(\rho)}_{A,a}\psi_{(\rho)})(x) = \rho(A)\psi_{(\rho)}\left(\Lambda(A)^{-1}(x+a)\right)\ , \qquad (39)$$

and the well-known basis-generators of the $s(M_0)$ algebra,

$$\hat{P}_\mu \equiv \hat{X}^{(\rho)}_{(\mu)} = i\partial_\mu\,, \tag{40}$$

$$\hat{J}^{(\rho)}_{\mu\nu} \equiv \hat{X}^{(\rho)}_{(\mu\nu)} = i(\eta_{\mu\alpha}x^\alpha\partial_\nu - \eta_{\nu\alpha}x^\alpha\partial_\mu) + S^{(\rho)}_{\mu\nu}\,, \tag{41}$$

which have point-independent spin parts denoted by $S^{(\rho)}_{\hat{\mu}\hat{\nu}}$ instead of $\rho(S_{\hat{\mu}\hat{\nu}})$. Hereby, it is convenient to denote the energy operator as $\hat{H} = \hat{P}_0$ and write the $sl(2,\mathbb{C})$ generators,

$$\hat{J}^{(\rho)}_i = \frac{1}{2}\varepsilon_{ijk}\hat{J}^{(\rho)}_{jk} = -i\varepsilon_{ijk}x^j\partial_k + S^{(\rho)}_i\,, \quad S^{(\rho)}_i = \frac{1}{2}\varepsilon_{ijk}S^{(\rho)}_{jk}\,, \tag{42}$$

$$\hat{K}^{(\rho)}_i = \hat{J}^{(\rho)}_{0i} = i(x^i\partial_t + t\partial_i) + S^{(\rho)}_{0i}\,, \qquad i,j,k... = 1,2,3\,, \tag{43}$$

denoting $\vec{S}^2 = S_iS_i$ and $\vec{S}^2_0 = S_{0i}S_{0i}$. Thus we lay out the standard basis of the $s(M_0)$ algebra, $\{\hat{H}, \hat{P}_i, \hat{J}^{(\rho)}_i, \hat{K}^{(\rho)}_i\}$.

The ***invariants*** of the manifest covariant fields are the eigenvalues of the Casimir operators of the reps. $T^{(\rho)}$ that read

$$\hat{C}_1 = \hat{P}_\mu\hat{P}^\mu\,, \quad \hat{C}^{(\rho)}_2 = -\eta_{\mu\nu}\hat{W}^{(\rho)\,\mu}\hat{W}^{(\rho)\,\nu}\,, \tag{44}$$

where the components of the Pauli-Lubanski operator [24],

$$\hat{W}^{(\rho)\,\mu} = -\frac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta}\hat{P}_\nu \hat{J}^{(\rho)}_{\alpha\beta}\,, \tag{45}$$

are defined by the skew-symmetric tensor with $\varepsilon^{0123} = -\varepsilon_{0123} = -1$. Thus we obtain,

$$\hat{W}_0^{(\rho)} = \hat{J}_i^{(\rho)}\hat{P}_i = S_i^{(\rho)}\hat{P}_i\,, \quad \hat{W}_i^{(\rho)} = \hat{H}\,\hat{J}_i^{(\rho)} + \varepsilon_{ijk}\hat{K}_j^{(\rho)}\hat{P}_k\,. \tag{46}$$

The first invariant (44a) gives the ***mass condition***, $\hat{P}^2\psi_{(\rho)} = m^2\psi_{(\rho)}$, fixing the orbit in the momentum spaces on which the fundamental solutions are defined. The second invariant is less relevant for the CRs since its form in configurations is quite complicated

$$\begin{aligned}\hat{\mathcal{C}}_2^{(\rho)} \;=\; & -(\vec{S}^{(\rho)})^2\partial_t^2 + 2(iS_{0k}^{(\rho)} - \varepsilon_{ijk}S_i^{(\rho)}S_{0j}^{(\rho)})\partial_k\partial_t \\ & - \left[(\vec{S}_0^{\,(\rho)})^2\Delta - (S_i^{(\rho)}S_j^{(\rho)} + S_{0i}^{(\rho)}S_{0j}^{(\rho)})\partial_i\partial_j\right]\,.\end{aligned} \tag{47}$$

Consequently, we may study its action in the momentum reps. where it selects the induced Wigner UIRs equivalent with the CR. Nevertheless, for fields with unique spin with $\rho = \rho(s) = (s,0)\oplus(0,s)$ we obtain in the rest frame where $P_i \sim 0$ that

$$\hat{\mathcal{C}}_2^{\rho(s)} = m^2 s(s+1) \tag{48}$$

since then $S_{0i}^{\rho(s)} = \pm i S_i^{\rho(s)}$.

In the Poincaré algebra we find the ***complete*** system of commuting operators $\{\hat{H}, \hat{P}_1, \hat{P}_2, \hat{P}_3\}$ defining the momentum rep.. The ***fundamental*** solutions are common eigenfunctions of this system such that any ***covariant*** quantum field can be written as

$$\psi_{(\rho)}(x) = \int d^3p \sum_{s\sigma} \left[U_{\vec{p},s\sigma}(x) a_{s\sigma}(\vec{p}) + V_{\vec{p},s\sigma}(x) b^{\dagger}_{s\sigma}(\vec{p}) \right] \tag{49}$$

where $a_{s\sigma}$ and $b_{s\sigma}$ are the field operators of a particle and antiparticle of ***spin*** s and ***polarization*** σ while the fundamental solutions have the form

$$U_{\vec{p},s\sigma}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} u_{s\sigma}(\vec{p}) e^{-iEt + i\vec{p}\cdot\vec{x}} , \quad V_{\vec{p},s\sigma}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} v_{s\sigma}(\vec{p}) e^{iEt - i\vec{p}\cdot\vec{x}} . \tag{50}$$

The vectors $u_{s\sigma}(\vec{p})$ and $v_{s\sigma}(\vec{p})$ have to be determined by the concrete form of the field equation and relativistic scalar product. However, when the field equations are linear we can postulate the orthonormalization relations

$$\overline{u}_{s\sigma}(\vec{p}) u_{s'\sigma'}(\vec{p}) = \overline{v}_{s\sigma}(\vec{p}) v_{s'\sigma'}(\vec{p}) = \delta_{ss'} \delta_{\sigma\sigma'} , \tag{51}$$

$$\overline{u}_{s\sigma}(\vec{p}) v_{s'\sigma'}(\vec{p}) = \overline{v}_{s\sigma}(\vec{p}) u_{s'\sigma'}(\vec{p}) = 0 . \tag{52}$$

that guarantee the separation of the particle and antiparticle sectors.

For the massive fields of mass m the momentum spans the orbit $\Omega_m = \{\vec{p}\,|\,p^2 = m^2\}$ which means that $p_0 = \pm E$ where $E = \sqrt{m^2 + \vec{p}^2}$. The solutions U are considered of ***positive*** frequencies having $p_0 = E$ while for the ***negative*** frequency ones, V, we must take $p_0 = -E$. In this manner the general rule (30) of separating the particle and antiparticle modes becomes

$$HU_{\vec{p},s\sigma} = EU_{\vec{p},s\sigma}\,, \quad HV_{\vec{p},s\sigma} = -EV_{\vec{p},s\sigma}\,, \tag{53}$$

$$\vec{P}U_{\vec{p},s\sigma} = \vec{p}\,U_{\vec{p},s\sigma}\,, \quad \vec{P}V_{\vec{p},s\sigma} = -\vec{p}\,V_{\vec{p},s\sigma}\,. \tag{54}$$

Wigner's induced UIRs

The Wigner theory of the induced UIRs is based on the fact that the orbits in momentum space may be built by using Lorentz transformations [6, 7]. In the case of massive particles we discuss here, any $\vec{p} \in \Omega_m$ can be obtained applying a ***boost*** transformation $L_{\vec{p}} \in L_+^\uparrow$ to the ***representative*** momentum $\mathring{p} = (m, 0, 0, 0)$ such that $\vec{p} = L_{\vec{p}}\,\mathring{p}$.

The rotations that leave $\overset{\circ}{p}$ invariant, $R\overset{\circ}{p} = \overset{\circ}{p}$, form the ***stable*** group $SO(3) \subset L_+^\uparrow$ whose universal covering group $SU(2)$ is called the ***little*** group associated to the representative momentum $\overset{\circ}{p}$.

We observe that the boosts $L_{\vec{p}}$ are defined up to a rotation since $L_{\vec{p}} R\,\overset{\circ}{p} = L_{\vec{p}}\overset{\circ}{p}$. Therefore, these span the homogeneous space $L_+^\uparrow/SO(3)$. The corresponding transformations of the $SL(2,\mathbb{C})$ group are denoted by $A_{\vec{p}} \in SL(2,\mathbb{C})/SU(2)$ assuming that these satisfy $\Lambda(A_{\vec{p}}) = L_{\vec{p}}$ and $A_{\overset{\circ}{p}} = 1 \in SL(2,\mathbb{C})$.

In applications one prefers to chose genuine Lorentz transformatios $A_{\vec{p}} = e^{-i\alpha n^i S_{0i}}$ with $\alpha = \mathrm{atrctanh}\frac{p}{E}$ and $n^i = \frac{p^i}{p}$ with $p = |\vec{p}|$. In the spinor rep. $\rho_s = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ of the Dirac theory one finds [27]

$$\rho_s(A_{\vec{p}}) = \frac{E + m + \gamma^0\gamma^i p^i}{\sqrt{2m(E+m)}}\,. \tag{55}$$

where γ^μ denote the Dirac matrices. The corresponding transformations of the $L_+^\uparrow$ group, $L_{\vec{p}} = \Lambda(A_{\vec{p}})$, have the matrix elements

$$(L_{\vec{p}})^{0\,\cdot}_{\cdot\,0} = \frac{E}{m}\,, \quad (L_{\vec{p}})^{0\,\cdot}_{\cdot\,i} = (L_{\vec{p}})^{i\,\cdot}_{\cdot\,0} = \frac{p^i}{m}\,, \quad (L_{\vec{p}})^{i\,\cdot}_{\cdot\,j} = \delta_{ij} + \frac{p^i p^j}{m(E+m)}\,. \tag{56}$$

Furthermore, we look for the transformations in momentum rep. generated by the CR under consideration. After a little calculation we obtain

$$\sum_{s'\sigma'} u_{s'\sigma'}(\vec{p})(T_{A,a}a_{s'\sigma'})(\vec{p}) = \sum_{s\sigma} \rho(A)u_{s\sigma}\left(\vec{p}'\right) a_{s\sigma}\left(\vec{p}'\right) e^{-ia\cdot p} \tag{57}$$

$$\sum_{s'\sigma'} v_{s'\sigma'}(\vec{p})(T_{A,a}b^{\dagger}_{s'\sigma'})(\vec{p}) = \sum_{s\sigma} \rho(A)v_{s\sigma}\left(\vec{p}'\right) b^{\dagger}_{s\sigma}\left(\vec{p}'\right) e^{ia\cdot p} \tag{58}$$

where $a\cdot p = Ea^0 - \vec{p}\cdot\vec{a}$ and $\vec{p}' = \Lambda(A)^{-1}\vec{p}$.

Focusing on the first equation, we introduce the ***Wigner mode functions***

$$u_{s\sigma}(\vec{p}) = \rho(A_{\vec{p}})\mathring{u}_{s\sigma} \tag{59}$$

where the vectors $\mathring{u}_{s\sigma} \in \mathcal{V}_{(\rho)}$ are independent on $\vec{p}$ and satisfy $\overline{\mathring{u}}_{s\sigma}\mathring{u}_{s'\sigma'} = \overline{u}_{s\sigma}(\vec{p})u_{s'\sigma'}(\vec{p}) = \delta_{ss'}\delta_{\sigma\sigma'}$ according to Eq. (51). We obtain thus the transformation rule of the Wigner reps. ***induced*** by the subgroup $T(4)Ⓢ SU(2)$ that read [6, 8]

$$\Rightarrow \quad (T_{A,a}a_{s\sigma})(\vec{p}) = \sum_{\sigma'} D^{s}_{\sigma\sigma'}(A,\vec{p})a_{s\sigma'}(\vec{p}')e^{i\vec{a}\cdot\vec{p}} \tag{60}$$

where

$$\Rightarrow \quad D^{s}_{\sigma\sigma'}(A,\vec{p}) = \overline{\mathring{u}}_{s\sigma}\rho[W(A,\vec{p})]\mathring{u}_{s\sigma'}\,, \quad W(A,\vec{p}) = A^{-1}_{\vec{p}}AA_{\vec{p}'} \tag{61}$$

The Wigner transformations $W(A,\vec{p})=A_{\vec{p}}^{-1}AA_{\vec{p}'}$ is of the little group $SU(2)$ since one can verify that $\Lambda[W(A,\vec{p})]=L_{\vec{p}}^{-1}\Lambda(A)L_{\vec{p}'}\in SO(3)$ leaving invariant the representative momentum $\mathring{p}$.

Therefore the matrices D^s realize the UIR of spin (s) of the little group $SU(2)$ that ***induces*** the Wigner UIR (60) denoted by $(s,\pm m)$ [8].

Note that the role of the vectors $\mathring{u}_{s\sigma}$ is ***to select*** the spin content of the CR determining the Wigner UIRs whose direct sum is equivalent to the CR $T^{(\rho)}$.

A similar procedure can be applied for the antiparticle but selecting the normalized vectors $\mathring{v}_{s\sigma}\in\mathcal{V}_{(\rho)}$ such that $\overline{\mathring{v}}_{s\sigma}\mathring{v}_{s'\sigma'}=\delta_{ss'}\delta_{\sigma\sigma'}$ and $\overline{\mathring{v}}_{s\sigma}\rho[W(A,\vec{p})]\mathring{v}_{s\sigma'}=D^s_{\sigma\sigma'}(A,\vec{p})^*$ since the operators a and b must transform alike under isometries [24]. Moreover, from Eq. (52) we deduce that the vectors $\mathring{u}$ and $\mathring{v}$ must be orthogonal, $\overline{\mathring{u}}_{s\sigma}\mathring{v}_{s'\sigma'}=\overline{\mathring{v}}_{s\sigma}\mathring{u}_{s'\sigma'}=0$.

The conclusion is that the CRs are ***equivalent*** to direct sums of Wigner UIRs with an arbitrary spin content defined by the vectors $\mathring{u}_{s\sigma}$ and $\mathring{v}_{s\sigma}$. For each spin s we meet the

UIR $(\pm m, s)$ in the space $\mathcal{V}_s \subset \mathcal{V}_{(\rho)}$ of the linear UIR of the group $SU(2)$ generated by the matrices $S_i^{(s)}$.

The transformation (60) allows us to derive the generators of the UIRs in momentum rep. (denoted by tilde) that are differential operator acting alike on the operators $a_{s\sigma}(\vec{p})$ and $b_{s\sigma}(\vec{p})$ seen as functions of $\vec{p}$. Thus for each UIR $(s, \pm m)$ we can write down the basis generators

$$\tilde{J}_i^{(s)} = -i\varepsilon_{ijk}p^j\partial_{p^k} + S_i^{(s)}\,, \tag{62}$$

$$\tilde{K}_i^{(s)} = iE\partial_{p^i} - \frac{p^i}{2E} + \frac{1}{E+m}\varepsilon_{ijk}p^j S_k^{(s)}\,. \tag{63}$$

With their help we derive the components of the Pauli-Lubanski operator

$$\tilde{W}_0^{(s)} = \vec{p}\cdot\vec{S}^{(s)}\,, \quad \tilde{W}_i^{(s)} = mS_i^{(s)} + \frac{p^i}{E+m}\vec{p}\cdot\vec{S}^{(s)}\,, \tag{64}$$

and we recover the well-known result [8]

$$\Rightarrow \quad \tilde{C}_1 = m^2\,, \quad \tilde{C}_2^{(s)} = m^2(\vec{S}^{(s)})^2 \sim m^2 s(s+1) \tag{65}$$

Finally we stress that the Wigner theory determine ***completely*** the form of the covariant fields ***without*** using field equations. Thus in special relativity we have two symmetric equivalent procedures:

(i) to start with the ***covariant field equation*** that gives the form of the covariant field determining its CR, or

(ii) to construct the ***Wigner covariant field*** and then to derive its field equation [24]. The typical example is the Dirac field in Minkowski spacetime [8, 27].

4. Covariant fields on de Sitter spacetime

The Wigner theory works only in local-Minkowskian manifold whose isometry group has a similar structure as the Poincaré one having an Abelian normal subgroup $T(4)$.

Unfortunately the Abelian group $T(3)_P$ of the de Sitter isometry group is not a normal (or invariant) subgroup such that the we must study of the de Sitter CRs in the configuration space following to consider the UIRs in momentum representation after the field is determined by a concrete field equation.

de Sitter isometries and Killing vectors

Let (M, g) be the de Sitter spacetime defined as the hyperboloid of radius $1/\omega$ in the five-dimensional flat spacetime (M^5, η^5) of coordinates z^A (labeled by the indices $A,\ B, ... = 0, 1, 2, 3, 4$) and metric $\eta^5 = \mathrm{diag}(1, -1, -1, -1, -1)$.

The local charts $\{x\}$ can be introduced on (M, g) giving the set of functions $z^A(x)$ which solve the hyperboloid equation,

$$\eta^5_{AB} z^A(x) z^B(x) = -\frac{1}{\omega^2} \, . \tag{66}$$

Here we use the chart $\{t, \vec{x}\}$ with the conformal time t and Cartesian spaces coordinates x^i defined by

$$\begin{aligned} z^0(x) &= -\frac{1}{2\omega^2 t}\left[1 - \omega^2(t^2 - \vec{x}^2)\right] \\ z^i(x) &= -\frac{1}{\omega t} x^i \, , \\ z^4(x) &= -\frac{1}{2\omega^2 t}\left[1 + \omega^2(t^2 - \vec{x}^2)\right] \end{aligned} \tag{67}$$

This chart covers the expanding part of M for $t \in (-\infty, 0)$ and $\vec{x} \in \mathbb{R}^3$ while the collapsing part is covered by a similar chart with $t > 0$. Both these charts have the conformal flat line element,

$$ds^2 = \eta^5_{AB} dz^A(x) dz^B(x) = \frac{1}{\omega^2 t^2}\left(dt^2 - d\vec{x}^2\right) \, . \tag{68}$$

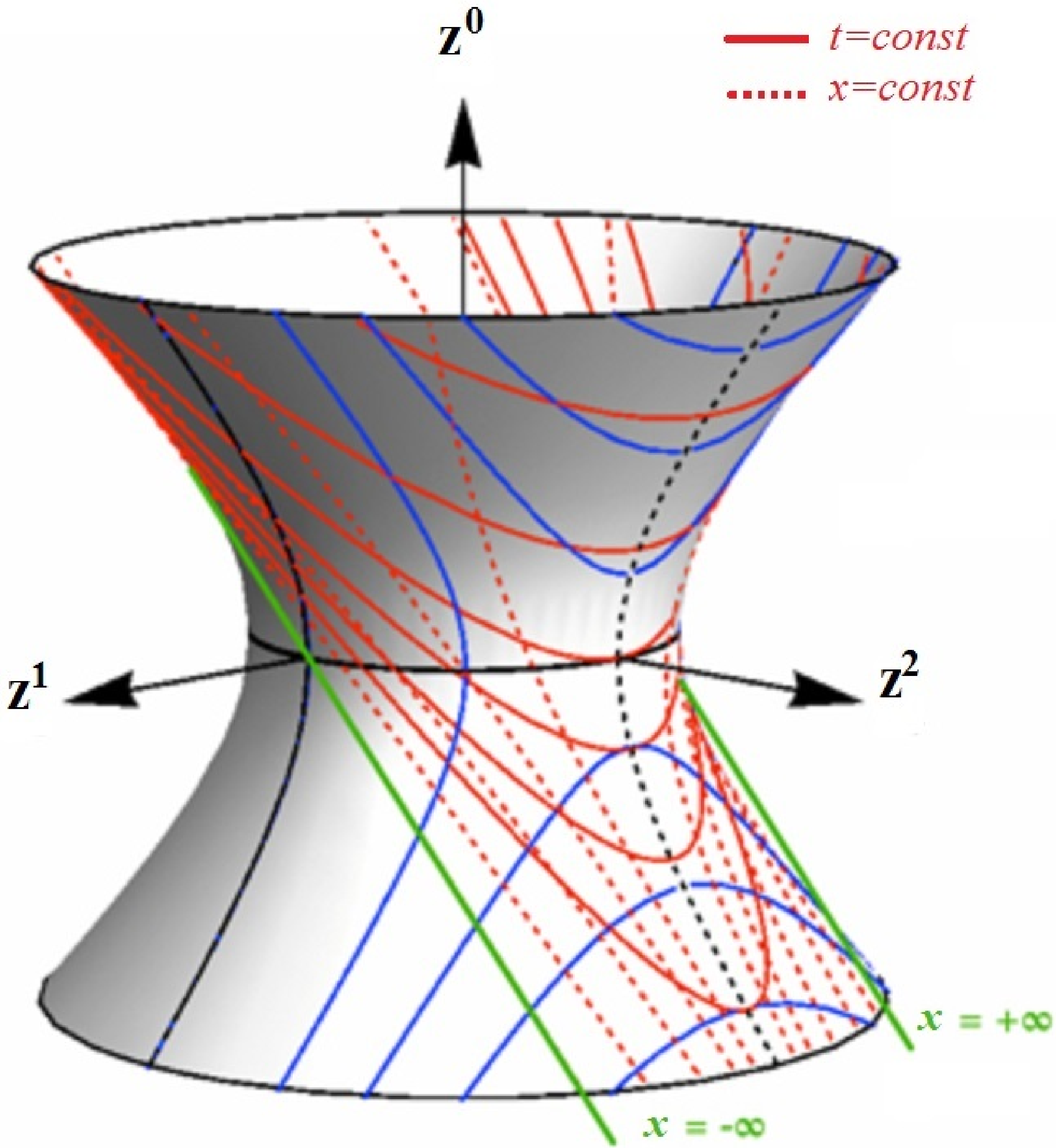

Figure 4: de Sitter spacetime.

In addition, we consider the local frames $\{t,\vec{x};e\}$ of the diagonal gauge,

$$e^0_0=-\omega t\,,\quad e^i_j=-\delta^i_j\,\omega t\,,\quad \hat{e}^0_0=-\frac{1}{\omega t}\,,\quad \hat{e}^i_j=-\delta^i_j\,\frac{1}{\omega t}\,. \tag{69}$$

The gauge group $G(\eta^5)=SO(1,4)$ is the isometry group of M, since its transformations, $z\to\mathfrak{g}z$, $\mathfrak{g}\in SO(1,4)$, leave the equation (66) invariant. Its universal covering group $\mathrm{Spin}(\eta^5)=Sp(2,2)$ is not involved directly in our construction since the spinor CRs are induced by the spinor representations of its subgroup $SL(2,\mathbb{C})$. Therefore, we can restrict ourselves to the group $SO(1,4)$ for which we adopt the parametrization

$$\mathfrak{g}(\xi)=\exp\left(-\frac{i}{2}\,\xi^{AB}\mathfrak{S}_{AB}\right)\in SO(1,4) \tag{70}$$

with skew-symmetric parameters, $\xi^{AB}=-\xi^{BA}$, and the covariant generators $\mathfrak{S}_{AB}$ of the fundamental representation of the $so(1,4)$ algebra carried by M^5. These generators have the matrix elements,

$$(\mathfrak{S}_{AB})^{C\,\cdot}_{\cdot\,D}=i\left(\delta^C_A\,\eta_{BD}-\delta^C_B\,\eta_{AD}\right)\,. \tag{71}$$

The principal $so(1,4)$ basis-generators with physical meaning [20] are the energy $\mathfrak{H} = \omega\mathfrak{S}_{04}$, angular momentum $\mathfrak{J}_k = \frac{1}{2}\varepsilon_{kij}\mathfrak{S}_{ij}$, Lorentz boosts $\mathfrak{K}_i = \mathfrak{S}_{0i}$, and the Runge-Lenz-type vector $\mathfrak{R}_i = \mathfrak{S}_{i4}$. In addition, it is convenient to introduce the momentum $\mathfrak{P}_i = -\omega(\mathfrak{R}_i + \mathfrak{K}_i)$ and its dual $\mathfrak{Q}_i = \omega(\mathfrak{R}_i - \mathfrak{K}_i)$ which are nilpotent matrices (i. e. $(\mathfrak{P}_i)^3 = (\mathfrak{Q}_i)^3 = 0$) generating two Abelian three-dimensional subgroups, $T(3)_P$ and respectively $T(3)_Q$. All these generators may form different bases of the algebra $so(1,4)$ as, for example, the basis $\{\mathfrak{H}, \mathfrak{P}_i, \mathfrak{Q}_i, \mathfrak{J}_i\}$ or the Poincaré-type one, $\{\mathfrak{H}, \mathfrak{P}_i, \mathfrak{J}_i, \mathfrak{K}_i\}$. We note that the four-dimensional restriction of the $so(1,3)$ subalegra generate the vector representation of the group $L^{\uparrow}_{+}$.

Using these generators we can derive the $SO(1,4)$ isometries, $\phi_{\mathfrak{g}}$, defined as

$$z[\phi_{\mathfrak{g}}(x)] = \mathfrak{g}\, z(x). \tag{72}$$

The transformations $\mathfrak{g} \in SO(3) \subset SO(4,1)$ generated by $\mathfrak{J}_i$, are simple rotations of z^i and x^i which transform alike since this symmetry is global. The transformations generated by $\mathfrak{H}$,

$$\exp(-i\xi\mathfrak{H}) \; : \quad \begin{array}{lcl} z^0 & \to & z^0\cosh\alpha - z^4\sinh\alpha \\ z^i & \to & z^i \\ z^4 & \to & -z^0\sinh\alpha + z^4\cosh\alpha \end{array} \tag{73}$$

whith $\alpha = \omega\xi$, produce the dilatations $t \to t\, e^{\alpha}$ and $x^i \to x^i e^{\alpha}$, while the $T(3)_P$ transformations

$$\exp(-i\xi^i \mathfrak{P}_i) \ : \quad \begin{array}{lcl} z^0 & \to & z^0 + \omega\, \vec{\xi} \cdot \vec{z} + \frac{1}{2}\, \omega^2 \vec{\xi}^{\,2} \, (z^0 + z^4) \\ z^i & \to & z^i + \omega\, \xi^i \, (z^0 + z^4) \\ z^4 & \to & z^4 - \omega\, \vec{\xi} \cdot \vec{z} - \frac{1}{2}\, \omega^2 \vec{\xi}^{\,2} \, (z^0 + z^4) \end{array} \tag{74}$$

give rise to the space translations $x^i \to x^i + \xi^i$ at fixed t. More interesting are the $T(3)_Q$ transformations generated by $\mathfrak{Q}_i/\omega$,

$$\exp(-i\xi^i \mathfrak{Q}_i/\omega) \ : \quad \begin{array}{lcl} z^0 & \to & z^0 - \vec{\xi} \cdot \vec{z} + \frac{1}{2}\, \vec{\xi}^{\,2} \, (z^0 - z^4) \\ z^i & \to & z^i - \xi^i \, (z^0 - z^4) \\ z^4 & \to & z^4 - \vec{\xi} \cdot \vec{z} + \frac{1}{2}\, \vec{\xi}^{\,2} \, (z^0 - z^4) \end{array} \tag{75}$$

which lead to the isometries

$$t \to \frac{t}{1 - 2\omega\, \vec{\xi} \cdot \vec{x} - \omega^2 \vec{\xi}^{\,2} \, (t^2 - \vec{x}^2)} \tag{76}$$

$$x^i \to \frac{x^i + \omega \xi^i \, (t^2 - \vec{x}^2)}{1 - 2\omega\, \vec{\xi} \cdot \vec{x} - \omega^2 \vec{\xi}^{\,2} \, (t^2 - \vec{x}^2)} \, . \tag{77}$$

We observe that $z^0 + z^4 = -\frac{1}{\omega^2 t}$ is invariant under translations (74), fixing the value of t, while $z^0 - z^4 = \frac{t^2 - \vec{x}^2}{t}$ is left unchanged by the $t(3)_Q$ transformations (75).

The orbital basis-generators of the natural representation of the $s(M)$ algebra (carried by the space of the scalar functions over M^5) have the standard form

$$L^5_{AB} = i\left[\eta^5_{AC} z^C \partial_B - \eta^5_{BC} z^C \partial_A\right] = -iK^C_{(AB)}\partial_C \tag{78}$$

which allows us to derive the corresponding Killing vectors of (M, g), $k_{(AB)}$, using the identities $k_{(AB)\mu} dx^\mu = K_{(AB)C} dz^C$. Thus we obtain the following components of the Killing vectors:

$$k^0_{(04)} = t\,, \quad k^i_{(04)} = x^i\,, \quad k^0_{(0i)} = k^0_{(4i)} = \omega t x^i \tag{79}$$

$$k^j_{(0i)} = \omega x^i x^j + \delta^j_i \frac{1}{2\omega}[\omega^2(t^2 - \vec{x}^2) - 1] \tag{80}$$

$$k^j_{(4i)} = \omega x^i x^j + \delta^j_i \frac{1}{2\omega}[\omega^2(t^2 - \vec{x}^2) + 1] \tag{81}$$

$$k^k_{(ij)} = \delta^k_j x^i - \delta^k_i x^j\,. \tag{82}$$

Generators of induced CRs

In the covariant parametrization of the $sp(2,2)$ algebra adopted here, the generators $X^{(\rho)}_{(AB)}$ corresponding to the Killing vectors $k_{(AB)}$ result from equation (21) and the functions (19) with the new labels $a \to (AB)$. Then we have

$$H = \omega X^{(\rho)}_{(04)} = -i\omega(t\partial_t + x^i\partial_i)\,, \tag{83}$$

$$J^{(\rho)}_i = \frac{1}{2}\varepsilon_{ijk}X^{(\rho)}_{(jk)} = -i\varepsilon_{ijk}x^j\partial_k + S^{(\rho)}_i\,, \quad S^{(\rho)}_i = \frac{1}{2}\varepsilon_{ijk}S^{(\rho)}_{jk}\,, \tag{84}$$

$$K^{(\rho)}_i = X^{(\rho)}_{(0i)} = x^i H + \frac{i}{2\omega}[1+\omega^2(\vec{x}^2 - t^2)]\partial_i - \omega t S^{(\rho)}_{0i} + \omega S^{(\rho)}_{ij}x^j\,, \tag{85}$$

$$R^{(\rho)}_i = X^{(\rho)}_{(i4)} = -K^{(\rho)}_i + \frac{1}{\omega}i\partial_i\,. \tag{86}$$

where H is the energy (or Hamiltonian), $\vec{J}$ total angular momentum , $\vec{K}$ generators of the Lorentz boosts , and $\vec{R}$ is a Runge-Lenz type vector. These generators form the basis $\{H, J^{(\rho)}_i, K^{(\rho)}_i, R^{(\rho)}_i\}$ of the covariant rep. of the $sp(2,2)$ algebra with the following commutation rules:

$$\left[J_i^{(\rho)}, J_j^{(\rho)}\right] = i\varepsilon_{ijk} J_k^{(\rho)}\,, \qquad \left[J_i^{(\rho)}, R_j^{(\rho)}\right] = i\varepsilon_{ijk} R_k^{(\rho)}\,, \tag{87}$$

$$\left[J_i^{(\rho)}, K_j^{(\rho)}\right] = i\varepsilon_{ijk} K_k^{(\rho)}\,, \qquad \left[R_i^{(\rho)}, R_j^{(\rho)}\right] = i\varepsilon_{ijk} J_k^{(\rho)}\,, \tag{88}$$

$$\left[K_i^{(\rho)}, K_j^{(\rho)}\right] = -i\varepsilon_{ijk} J_k^{(\rho)}\,, \qquad \left[R_i^{(\rho)}, K_j^{(\rho)}\right] = \frac{i}{\omega}\,\delta_{ij} H\,, \tag{89}$$

and

$$\left[H, J_i^{(\rho)}\right] = 0\,, \quad \left[H, K_i^{(\rho)}\right] = i\omega R_i^{(\rho)}\,, \quad \left[H, R_i^{(\rho)}\right] = i\omega K_i^{(\rho)}\,. \tag{90}$$

In some applications it is useful to replace the operators $\vec{K}^{(\rho)}$ and $\vec{R}^{(\rho)}$ by the Abelian ones, i. e. the momentum operator $\vec{P}$ and its dual $\vec{Q}^{(\rho)}$, whose components are defined as

$$P_i = -\omega(R_i^{(\rho)} + K_i^{(\rho)}) = -i\partial_i\,, \quad Q_i^{(\rho)} = \omega(R_i^{(\rho)} - K_i^{(\rho)})\,, \tag{91}$$

obtaining the new basis $\{H, P_i, Q_i^{(\rho)}, J_i^{(\rho)}\}$.

The last two bases bring together the conserved energy (83) and momentum (91a) which are the only genuine orbital operators, independent on ρ. What is specific for the de Sitter symmetry is that these operators can not be put simultaneously in diagonal form since they do not commute to each other.

Casimir operators

The first invariant of the CR $T^{(\rho)}$ is the quadratic Casimir operator

$$\begin{aligned} \mathcal{C}_1^{(\rho)} &= -\omega^2 \frac{1}{2} X^{(\rho)}_{(AB)} X^{(\rho)\,(AB)} && (92) \\ &= H^2 + 3i\omega H - \vec{Q}^{(\rho)} \cdot \vec{P} - \omega^2 \vec{J}^{(\rho)} \cdot \vec{J}^{(\rho)} \,. && (93) \end{aligned}$$

After a few manipulation we obtain its definitive expression

$$\mathcal{C}_1^{(\rho)} = \mathcal{E}_{KG} + 2i\omega e^{-\omega t} S^{(\rho)}_{0i} \partial_i - \omega^2 (\vec{S}^{(\rho)})^2 \,, \tag{94}$$

depending on the Klein-Gordon operator $\mathcal{E}_{KG} = -\partial_t^2 - 3\,\omega \partial_t + e^{-2\omega t}\Delta$.

The second Casimir operator, $\mathcal{C}_2^{(\rho)} = -\eta^5_{AB} W^{(\rho)\,A} W^{(\rho)\,B}$, is written with the help of the five-dimensional vector-operator $W^{(\rho)}$ whose components read [13]

$$W^{(\rho)\,A} = \frac{1}{8}\,\omega\,\varepsilon^{ABCDE} X^{(\rho)}_{(BC)} X^{(\rho)}_{(DE)} \,, \tag{95}$$

where $\varepsilon^{01234} = 1$ and the factor ω assures the correct flat limit. After a little calculation we obtain the concrete form of these components,

$$
\begin{aligned}
W_0^{(\rho)} &= \omega\, \vec{J}^{(\rho)} \cdot \vec{R}^{(\rho)}\,, && (96)\\
W_i^{(\rho)} &= H\, J_i^{(\rho)} + \omega\, \varepsilon_{ijk} K_j^{(\rho)} R_k^{(\rho)}\,, && (97)\\
W_4^{(\rho)} &= -\omega\, \vec{J}^{(\rho)} \cdot \vec{K}^{(\rho)}\,, && (98)
\end{aligned}
$$

which indicate that $W^{(\rho)}$ plays an important role in theories with spin, similar to that of the Pauli-Lubanski operator (45) of the Poincaré symmetry. For example, the helicity operator is now $W_0^{(\rho)} - W_4^{(\rho)} = S_i^{(\rho)} P_i$.

Then by using the components (96)-(98) we are faced with a complicated calculation but which can be performed using algebraic codes on computer. Thus we obtain the closed form of the second Casimir operator,

$$
\begin{aligned}
\mathcal{C}_2^{(\rho)} = & -\omega^2 (\vec{S}^{(\rho)})^2 (t^2 \partial_t^2 - 2t\partial_t + 2) + 2\omega^2 t^2 (i S_{0k}^{(\rho)} - \varepsilon_{ijk} S_i^{(\rho)} S_{0j}^{(\rho)}) \partial_k \partial_t \\
& + \omega t \left[(\vec{S_0}^{(\rho)})^2 \Delta - (S_i^{(\rho)} S_j^{(\rho)} + S_{0i}^{(\rho)} S_{0j}^{(\rho)}) \partial_i \partial_j \right] \\
& - 2i\omega^2 t (S_i^{(\rho)} S_k^{(\rho)} S_{0i}^{(\rho)} + S_{0k}^{(\rho)}) \partial_k\,. \qquad (99)
\end{aligned}
$$

In the case of fields with unique spin s we must select the reps. $\rho(s)=(s,0)\oplus(0,s)$, for which we have to replace $S_{0i}^{\rho(s)}=\pm\, iS_i^{\rho(s)}$ in equation (99) finding the remarkable identity

$$\mathcal{C}_2^{\rho(s)}=\mathcal{C}_1^{\rho(s)}(\vec{S}^{\rho(s)})^2-2\omega^2(\vec{S}^{\rho(s)})^2+\omega^2[(\vec{S}^{\rho(s)})^2]^2\,. \tag{100}$$

It is interesting to look for the invariants of the particles at rest in the chart $\{t,\vec{x}\}$. These have the vanishing momentum ($P_i\sim 0$) so that H acts as $i\partial_t$ and, therefore, it can be put in diagonal form its eigenvalues being just the rest energies, E_0. Then, for each subspace $\mathcal{V}_s\subset\mathcal{V}_{(\rho)}$ of given spin, s, we obtain the eigenvalues of the first Casimir operator,

$$\mathcal{C}_1^{\rho(s)}\sim E_0^2+3i\omega E_0-\omega^2 s(s+1)\,, \tag{101}$$

using Eq. (94) while those of the second Casimir operator,

$$\mathcal{C}_2^{\rho(s)}\sim s(s+1)(E_0^2+3i\omega E_0-2\omega^2)\,, \tag{102}$$

result from equation (99). These eigenvalues are real numbers so that the rest energies, $E_0=\Re E_0-\frac{3i\omega}{2}$, must be complex numbers whose imaginary parts are due to the decay produced by the de Sitter expansion.

The above results indicate that ***the CRs are reducible to direct sums of UIRs of the principal series*** [9]. These are labeled by two weights, (p,q), with $p=s$ while q is a solution of the equation $q(1-q)=\frac{1}{\omega^2}\left(\Re E_0\right)^2+\frac{1}{4}$.

In the ***flat limit*** we recover the Poincaré generators. We observe that the generators (84) are independent on ω having the same form as in the Minkowski case, $J_k^{(\rho)}=\hat{J}_k^{(\rho)}$. The other generators have the limits

$$\lim_{\omega\to 0} H=\hat{H}=i\partial_t\,,\quad \lim_{\omega\to 0}(\omega R_i^{(\rho)})=-\hat{P}_i=i\partial_i\,,\quad \lim_{\omega\to 0} K_i^{(\rho)}=\hat{K}_i^{(\rho)}\,,\qquad (103)$$

which means that the basis $\{H,\,P_i,\,J_i^{(\rho)},\,K_i^{(\rho)}\}$ of the algebra $s(M)=sp(2,2)$ tends to the basis $\{\hat{H},\,\hat{P}_i,\,\hat{J}_i^{(\rho)},\,\hat{K}_i^{(\rho)}\}$ of the $s(M_0)$ algebra when $\omega\to 0$. Moreover, the Pauli-Lubanski operator (45) is the flat limit of the five-dimensional vector-operator (95) since

$$\lim_{\omega\to 0} W_0^{(\rho)}=\hat{W}_0^{(\rho)}\,,\quad \lim_{\omega\to 0} W_i^{(\rho)}=\hat{W}_i^{(\rho)}\,,\quad \lim_{\omega\to 0} W_4^{(\rho)}=0\,.\qquad (104)$$

Under such circumstances the limits of our invariants read

$$\lim_{\omega\to 0}\mathcal{C}_1^{(\rho)}=\hat{\mathcal{C}}_1=\hat{P}^2\,,\quad \lim_{\omega\to 0}\mathcal{C}_2^{(\rho)}=\hat{\mathcal{C}}_2^{(\rho)}\,,\qquad (105)$$

indicating that their physical meaning may be related to the mass and spin of the matter fields in a similar manner as in special relativity.

	Minkowski		de Sitter
CRs	$manifest\ SL(2,\mathbb{C})$	**CRs**	$induced\ by\ SL(2,\mathbb{C})$
H	$= i\partial_t$	H	$= -i\omega(t\partial_t + x^i\partial_i)$
P^i	$= -i\partial_i$	P^i	$= -i\partial_i$
$J_i^{(\rho)}$	$= -i\varepsilon_{ijk}x^j\partial_k + S_i^{(\rho)}$	$J_i^{(\rho)}$	$= -i\varepsilon_{ijk}x^j\partial_k + S_i^{(\rho)}$
$K_i^{(\rho)}$	$= i(x^i\partial_t + t\partial_i) + S_{0i}^{(\rho)}$	$K_i^{(\rho)}$	$= x^i H + \frac{i}{2\omega}[1+\omega^2(\vec{x}^2 - t^2)]\partial_i - \omega t S_{0i}^{(\rho)} + \omega S_{ij}^{(\rho)} x^j$
UIR	$\tilde{\mathcal{P}}_+^\uparrow = T(4)Ⓢ SL(2,\mathbb{C})$	**UIR**	$\mathrm{Spin}(1,4) = Sp(2,2)$
$\mathcal{C}_1$	$= m^2$	$\mathcal{C}_1^{\rho(s)}$	$= M^2 + \frac{9}{4}\omega^2 - \omega^2 s(s+1)$
$\mathcal{C}_2^{\rho(s)}$	$= m^2 s(s+1)$	$\mathcal{C}_2^{\rho(s)}$	$= \left(M^2 + \frac{1}{4}\omega^2\right) s(s+1)$
	where $m = E_0$		where $M = \Re E_0$

Scalar field	$s = 0$	$M = \sqrt{m^2 - \frac{9}{4}\omega^2}$	$\mathcal{C}_1 = m^2$	$\mathcal{C}_2 = 0$
Dirac field	$s = \frac{1}{2}$	$M = m$	$\mathcal{C}_1 = m^2 + \frac{3}{2}\omega^2$	$\mathcal{C}_2 = \frac{3}{4}\left(m^2 + \frac{1}{4}\omega^2\right)$
Proca field	$s = 1$	$M = \sqrt{m^2 - \frac{1}{4}\omega^2}$	$\mathcal{C}_1 = m^2$	$\mathcal{C}_2 = 2m^2$

5. The Dirac field on de Sitter spacetimes

In the absence of a strong theory like the Wigner one in the flat case we must study the CR-UIR equivalence resorting to the covariant field equations able to give us the structure of the covariant field. Then, bearing in mind that the de Sitter UIRs are well-studied [9, 10], we can establish the CR-UIR equivalence by studying the CR and UIR Casimir operators in configurations and momentum rep..

In what follows we concentrate on the Dirac equation on the de Sitter spacetime since this is the only equation on this background giving the natural rest energy $\Re E_0 = m$ [20].

Invariants of the spinor CR

In the frame $\{t, \vec{x}; e\}$ introduced above the free Dirac equation takes the form [18],

$$(\mathcal{E}_D - m)\psi(x) = \left[-i\omega t\left(\gamma^0\partial_t + \gamma^i\partial_i\right) + \frac{3i\omega}{2}\gamma^0 - m\right]\psi(x) = 0\,, \qquad (106)$$

depending on the point-independent Dirac matrices $\gamma^{\hat\mu}$ that satisfy $\{\gamma^{\hat\alpha},\, \gamma^{\hat\beta}\} = 2\eta^{\hat\alpha\hat\beta}$ giving rise to the basis-generators $S^{(\rho_s)\,\hat\alpha\hat\beta} = \frac{i}{4}[\gamma^{\hat\alpha}, \gamma^{\hat\beta}]$ of the spinor rep. $\rho_s = \rho(\frac{1}{2}) = (\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$ of the group $\hat G = SL(2, \mathbb{C})$ that induces the spinor CR [3, 18, 19].

Eq. (106) can be analytically solved either in momentum or energy bases with correct orthonormalization and completeness properties [18, 19] with respect to the relativistic scalar product

$$\langle\psi, \psi'\rangle = \int d^3x\, (-\omega t)^{-3}\overline{\psi}(t, \vec x)\gamma^0\psi'(t, \vec x)\,. \tag{107}$$

The mode expansion in the spin-momentum rep. [19],

$$\psi(t, \vec x) = \int d^3p \sum_\sigma \left[U_{\vec p,\sigma}(x)a(\vec p, \sigma) + V_{\vec p,\sigma}(x)b^\dagger(\vec p, \sigma)\right]\ , \tag{108}$$

is written in terms of the field operators, a and b (satisfying canonical anti-commutation rules), and the particle and antiparticle fundamental spinors of momentum $\vec p$ (with $p = |\vec p|$) and polarization $\sigma = \pm\frac{1}{2}$,

$$U_{\vec p,\sigma}(t, \vec x\,) = \frac{1}{(2\pi)^{\frac{3}{2}}}\, u_{\vec p,\sigma}(t)e^{i\vec p\cdot\vec x}\,, \quad V_{\vec p,\sigma}(t, \vec x\,) = \frac{1}{(2\pi)^{\frac{3}{2}}}\, v_{\vec p,\sigma}(t)e^{-i\vec p\cdot\vec x} \tag{109}$$

whose time-dependent terms have the form [19, 21]

$$u_{\vec{p},\sigma}(t) = \frac{i}{2}\left(\frac{\pi p}{\omega}\right)^{\frac{1}{2}}(\omega t)^2 \left(\begin{array}{r} e^{\frac{1}{2}\pi\mu} H^{(1)}_{\nu_-}(-pt)\, \xi_\sigma \\ e^{-\frac{1}{2}\pi\mu} H^{(1)}_{\nu_+}(-pt)\, \frac{\vec{\sigma}\cdot\vec{p}}{p}\, \xi_\sigma \end{array}\right), \tag{110}$$

$$v_{\vec{p},\sigma}(t) = \frac{i}{2}\left(\frac{\pi p}{\omega}\right)^{\frac{1}{2}}(\omega t)^2 \left(\begin{array}{r} e^{-\frac{1}{2}\pi\mu} H^{(2)}_{\nu_-}(-pt)\, \frac{\vec{\sigma}\cdot\vec{p}}{p}\, \eta_\sigma \\ e^{\frac{1}{2}\pi\mu} H^{(2)}_{\nu_+}(-pt)\, \eta_\sigma \end{array}\right), \tag{111}$$

in the standard rep. of the Dirac matrices (with diagonal γ^0) and a fixed vacuum of the Bounch-Davies type [21]. Obviously, the notation σ_i stands for the Pauli matrices while the point-independent Pauli spinors ξ_σ and $\eta_\sigma = i\sigma_2(\xi_\sigma)^*$ are normalized as $\xi^+_\sigma \xi_{\sigma'} = \eta^+_\sigma \eta_{\sigma'} = \delta_{\sigma\sigma'}$ [19]. The terms giving the time modulation depend on the Hankel functions $H^{(1,2)}_{\nu_\pm}$ of indices

$$\nu_\pm = \frac{1}{2} \pm i\mu\,, \quad \mu = \frac{m}{\omega}\,. \tag{112}$$

Based on their properties (presented in Appendix B) we deduce

$$u^+_{\vec{p},\sigma}(t) u_{\vec{p},\sigma}(t) = v^+_{\vec{p},\sigma}(t) v_{\vec{p},\sigma}(t) = (-\omega t)^3 \tag{113}$$

obtaining the ortonormalization relations [18]

$$\langle U_{\vec{p},\sigma}, U_{\vec{p}',\sigma'}\rangle = \langle V_{\vec{p},\sigma}, V_{\vec{p}',\sigma'}\rangle = \delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\,, \tag{114}$$

$$\langle U_{\vec{p},\sigma}, V_{\vec{p}',\sigma'}\rangle = \langle V_{\vec{p},\sigma}, U_{\vec{p}',\sigma'}\rangle = 0\,, \tag{115}$$

that yield the useful inversion formulas, $a(\vec{p},\sigma) = \langle U_{\vec{p},\sigma}, \psi\rangle$ and $b(\vec{p},\sigma) = \langle \psi, V_{\vec{p},\sigma}\rangle$. Moreover, it is not hard to verify that these spinors are charge-conjugated to each other,

$$V_{\vec{p},\sigma} = (U_{\vec{p},\sigma})^c = \mathcal{C}(\overline{U}_{\vec{p},\sigma})^T\,, \quad \mathcal{C} = i\gamma^2\gamma^0\,, \tag{116}$$

and represent a ***complete*** system of solutions in the sense that [18]

$$\int d^3p \sum_{\sigma}\left[U_{\vec{p},\sigma}(t,\vec{x})U^{+}_{\vec{p},\sigma}(t,\vec{x}\,') + V_{\vec{p},\sigma}(t,\vec{x})V^{+}_{\vec{p},\sigma}(t,\vec{x}\,')\right] = e^{-3\omega t}\delta^3(\vec{x}-\vec{x}\,')\,. \tag{117}$$

The Dirac field transforms under isometries $x \to x' = \phi_{\mathfrak{g}}(x)$ (with $\mathfrak{g} \in I(M)$) according to the CR $T_{\mathfrak{g}} : \psi(x) \to (T_{\mathfrak{g}}\psi)(x') = A_{\mathfrak{g}}(x)\psi(x)$ whose generators are given by Eqs (83) - (86) where now $\rho = \rho_s$. Then, according to equations (94) and (106) we obtain the identity

$$\mathcal{C}_1^{(\rho_s)} = \mathcal{E}_D^2 + \frac{3}{2}\omega^2\mathbf{1}_{4\times 4} \sim m^2 + \frac{3}{2}\omega^2\,. \tag{118}$$

This result and equation (101) yield the rest energy of the Dirac field,

$$E_0 = -\frac{3i\omega}{2} \pm m\,, \tag{119}$$

which has a natural simple form where the decay (first) term is added to the usual rest energy of special relativity. A similar result can be obtained by solving the Dirac equation with vanishing momentum.

The second invariant results from equations (100) and (118) if we take into account that $(\vec{S}^{(\rho_s)})^2 = \frac{3}{4}\,\mathbf{1}_{4\times4}$. Thus we find

$$\mathcal{C}_2^{(\rho_s)} = \frac{3}{4}\,\mathcal{E}_D^2 + \frac{3}{16}\,\omega^2\mathbf{1}_{4\times4} \sim \frac{3}{4}\left(m^2 + \frac{1}{4}\,\omega^2\right) = \omega^2 s(s+1)\nu_+\nu_-\,, \tag{120}$$

where $s = \frac{1}{2}$ is the spin and $\nu_\pm = \frac{1}{2} \pm i\frac{m}{\omega}$ are the indices of the Hankel functions giving the time modulation of the Dirac spinors of the momentum basis [18].

These invariants define the UIRs that in the flat limit become Wigner's UIRs $(\pm m, \frac{1}{2})$ since

$$\lim_{\omega\to0} \mathcal{C}_1^{(\rho_s)} \sim m^2\,, \quad \lim_{\omega\to0} \mathcal{C}_2^{(\rho_s)} \sim \frac{3}{4}\,m^2\,. \tag{121}$$

Invariants of UIRs in momentum rep.

The above inversion formulas allow us to write the transformation rules in momentum rep. as

$$(T_{\mathfrak{g}}a)(\vec{p},\sigma) = \left\langle U_{\vec{p},\sigma}, [\rho_s(A_{\mathfrak{g}})\psi] \circ \phi_{\mathfrak{g}}^{-1} \right\rangle , \tag{122}$$

$$(T_{\mathfrak{g}}b)(\vec{p},\sigma) = \left\langle [\rho_s(A_{\mathfrak{g}})\psi] \circ \phi_{\mathfrak{g}}^{-1}, V_{\vec{p},\sigma} \right\rangle , \tag{123}$$

but, unfortunately, these scalar product are complicated integrals that cannot be solved. Therefore, we must restrict ourselves to study the corresponding Lie algebras focusing on the basis generators in momentum rep..

Any self-adjoint generator X of the spinor rep. of the group $S(M)$ gives rise to a ***conserved*** one-particle operator of the QFT,

$$\mathbf{X} =: \langle \psi, X\psi \rangle := \mathbf{X}^{(+)} + \mathbf{X}^{(-)} = \int d^3p \left[\alpha^\dagger(\vec{p})\tilde{X}^{(+)}\alpha(\vec{p}) + \beta^\dagger(\vec{p})\tilde{X}^{(-)}\beta(\vec{p}) \right] , \tag{124}$$

calculated respecting the normal ordering of the operator products [26]. The operators $\tilde{X}^{(\pm)}$ are the generators of CRs in momentum rep. acting on the operator valued Pauli spinors,

$$\alpha(\vec{p}) = \begin{pmatrix} a(\vec{p}, \frac{1}{2}) \\ a(\vec{p}, -\frac{1}{2}) \end{pmatrix}, \quad \beta(\vec{p}) = \begin{pmatrix} b(\vec{p}, \frac{1}{2}) \\ b(\vec{p}, -\frac{1}{2}) \end{pmatrix}. \tag{125}$$

As observed in Ref. [16], the straightforward method for finding the structure of these operators is to evaluate the entire expression (124) by using the form (108) where the field operators a and b satisfy the ***canonical anti-commutation rules*** [16, 18].

For this purpose we consider several identities written with the notation $\partial_{p_i} = \frac{\partial}{\partial p_i}$ as

$$H\, U_{\vec{p},\sigma}(t,\vec{x}) = -i\omega \left(p^i \partial_{p^i} + \frac{3}{2}\right) U_{\vec{p},\sigma}(t,\vec{x})\,,$$

$$H\, V_{\vec{p},\sigma}(t,\vec{x}) = -i\omega \left(p^i \partial_{p^i} + \frac{3}{2}\right) V_{\vec{p},\sigma}(t,\vec{x})\,,$$

that help us to eliminate some multiplicative operators and the time derivative when we inverse the Fourier transform. Furthermore, by applying the Green theorem and calculating some terms on computer we find two ***identical*** reps. whose basis generators read, $\tilde{P}_i^{(\pm)} = \tilde{P}_i = p_i$ and

$$\tilde{H}^{(\pm)} = \omega \tilde{X}^{(\pm)}_{(04)} = i\omega \left(p_i \partial_{p_i} + \frac{3}{2} \right) \tag{126}$$

$$\tilde{J}_i^{(\pm)} = \frac{1}{2} \varepsilon_{ijk} \tilde{X}^{(\pm)}_{(jk)} = -i\varepsilon_{ijk} p_j \partial_{p_k} + \frac{1}{2}\sigma_i \tag{127}$$

$$\tilde{K}_i^{(\pm)} = \tilde{X}^{(\pm)}_{(0i)} = i\tilde{H}^{(\pm)}\partial_{p_i} + \frac{\omega}{2} p_i \Delta_p - p_i \frac{\vec{p}^2 + m^2}{2\omega \vec{p}^2} + \frac{1}{2}\varepsilon_{ijk} \left(i\omega \partial_{p_j} - p_j \frac{m}{2\vec{p}^2} \right) \sigma_k \tag{128}$$

$$\tilde{R}_i^{(\pm)} = \tilde{X}^{(\pm)}_{(i4)} = -\tilde{K}_i^{(\pm)} - \frac{1}{\omega}\tilde{P}_i \,, \tag{129}$$

where $\Delta_p = \partial_{p_i}\partial_{p_i}$. These basis generators satisfy the specific $sp(2,2)$ commutation rules of the form (87)-(90). Moreover, it is not difficult to verify that these are ***Hermitian*** operators with respect to the scalar products of the momentum rep.

$$\langle \alpha, \alpha' \rangle = \int d^3p \, \alpha^\dagger(\vec{p}) \tilde{\alpha}(\vec{p}) \,, \quad \langle \beta, \beta' \rangle = \int d^3p \beta^\dagger(\vec{p}) \tilde{\beta}(\vec{p}) \,. \tag{130}$$

Therefore, we can conclude that these operators generate a pair of ***unitary*** reps. of the group $S(M)$.

Starting with the Pauli-Lubanski operator,

$$\tilde{W}_0^{(\pm)} = \frac{\omega}{4}(\vec{\sigma}\cdot\vec{p})\Delta_p + \frac{\omega\nu_-}{2}\vec{\sigma}\cdot\vec{\partial}_p + \frac{im}{2p^2}(\vec{\sigma}\cdot\vec{p})\,\vec{p}\cdot\vec{\partial}_p + \frac{m^2-\vec{p}^2+2i\omega m}{4\vec{p}^2\omega}\vec{\sigma}\cdot\vec{p}\,, \tag{131}$$

$$\tilde{W}_i^{(\pm)} = \frac{i}{2}(\vec{\sigma}\cdot\vec{p})\partial_{p_i} - \frac{i\nu_-}{2\vec{p}^2}\sigma_i - \frac{m}{2\omega\vec{p}^2}(\vec{\sigma}\cdot\vec{p})p_i\,, \tag{132}$$

$$\tilde{W}_4^{(\pm)} = \tilde{W}_0^{(\pm)} + \frac{1}{2\omega}\vec{\sigma}\cdot\vec{p}\,, \tag{133}$$

we calculate on computer the following Casimir operators,

$$\tilde{\mathcal{C}}_1^{(\pm)} = \omega^2[-s(s+1)-(q+1)(q-2)] = m^2+\frac{3\omega^2}{2}\,, \tag{134}$$

$$\tilde{\mathcal{C}}_2^{(\pm)} = \omega^2[-s(s+1)q(q-1)] = \omega^2 s(s+1)\nu_+\nu_- = \frac{3}{4}\left(m^2+\frac{\omega^2}{4}\right)\,, \tag{135}$$

recovering thus the results (118) and (120) obtained in configurations.

	Dirac field on Minkowski	Dirac field on de Sitter
UIR	$(\frac{1}{2}, \pm m)\ of\ \tilde{\mathcal{P}}_+^\uparrow$	$(\frac{1}{2}, \nu_\pm)\ of\ Sp(2,2)$

Dirac field on Minkowski:

$$
\begin{aligned}
P^i &= p^i \\
\tilde{H} &= E = \sqrt{m^2 + \vec{p}^2} \\
\tilde{J}_i^{(\pm)} &= -i\varepsilon_{ijk} p_j \partial_{p_k} + \tfrac{1}{2}\sigma_i \\
\tilde{K}_i^{(\pm)} &= iE\partial_{p^i} - \frac{p^i}{2E} \\
&\quad + \frac{1}{2(E+m)}\,\varepsilon_{ijk} p^j \sigma_k
\end{aligned}
$$

$$
\begin{aligned}
\tilde{W}_0^{(\pm)} &= \tfrac{1}{2}\,\vec{p}\cdot\vec{\sigma} \\
\tilde{W}_i^{(\pm)} &= \tfrac{1}{2}\,m\sigma_i + \frac{p^i}{2(E+m)}\vec{\sigma}\cdot\vec{p}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathcal{C}}_1^{(\pm)} &= m^2 \\
\tilde{\mathcal{C}}_2^{(\pm)} &= \tfrac{3}{4}\,m^2
\end{aligned}
$$

Dirac field on de Sitter:

$$
\begin{aligned}
P^i &= p^i \\
\tilde{H}^{(\pm)} &= i\omega\left(p_i\partial_{p_i} + \tfrac{3}{2}\right) \\
\tilde{J}_i^{(\pm)} &= -i\varepsilon_{ijk} p_j \partial_{p_k} + \tfrac{1}{2}\sigma_i \\
\tilde{K}_i^{(\pm)} &= i\tilde{H}^{(\pm)}\partial_{p_i} + \tfrac{\omega}{2} p_i \Delta_p - p_i \frac{\vec{p}^2 + m^2}{2\omega\vec{p}^2} \\
&\quad + \tfrac{1}{2}\varepsilon_{ijk}\left(i\omega\partial_{p_j} - p_j\frac{m}{2\vec{p}^2}\right)\sigma_k
\end{aligned}
$$

$$
\begin{aligned}
\tilde{W}_0^{(\pm)} &= \tfrac{\omega}{4}(\vec{\sigma}\cdot\vec{p})\Delta_p + \frac{\omega\nu_-}{2}\vec{\sigma}\cdot\vec{\partial}_p \\
&\quad + \frac{im}{2p^2}(\vec{\sigma}\cdot\vec{p})\,\vec{p}\cdot\vec{\partial}_p + \frac{m^2 - \vec{p}^2 + 2i\omega m}{4\vec{p}^2\omega}\vec{\sigma}\cdot\vec{p} \\
\tilde{W}_i^{(\pm)} &= \tfrac{i}{2}(\vec{\sigma}\cdot\vec{p})\partial_{p_i} - \frac{i\nu_-}{2\vec{p}^2}\sigma_i - \frac{m}{2\omega\vec{p}^2}(\vec{\sigma}\cdot\vec{p})p_i \\
\tilde{W}_4^{(\pm)} &= \tilde{W}_0^{(\pm)} + \frac{1}{2\omega}\vec{\sigma}\cdot\vec{p}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathcal{C}}_1^{(\pm)} &= m^2 + \tfrac{3}{2}\,\omega^2 \\
\tilde{\mathcal{C}}_2^{(\pm)} &= \tfrac{3}{4}\left(m^2 + \tfrac{1}{4}\,\omega^2\right)
\end{aligned}
$$

CR-UIR equivalence

The above result shows that the identical spinor reps. we obtained here are UIRs of the principal series corresponding to the canonical labels (s,q) with $s=\frac{1}{2}$ and $q=\nu_{\pm}$. In other words the spinor CR of the Dirac theory is equivalent with the orthogonal sum of the equivalent UIRs of the particle and antiparticle sectors.

This suggests that the UIRs $(s,\nu_{\pm})$ of the group $S(M) = Sp(2,2)$ can be seen as being analogous to the Wigner ones $(s,\pm m)$ of the Dirac theory in Minkowski spacetime.

In general, the above equivalent spinor UIRs may not coincide since the expressions of their basis generators are strongly dependent on the arbitrary phase factors of the fundamental spinors whether these depend on $\vec{p}$. Thus if we change

$$U_{\vec{p},\sigma}\to e^{i\chi^{+}(\vec{p})}U_{\vec{p},\sigma}\,,\quad V_{\vec{p},\sigma}\to e^{-i\chi^{-}(\vec{p})}V_{\vec{p},\sigma}\,, \tag{136}$$

with $\chi^{\pm}(\vec{p})\in\mathbb{R}$, performing simultaneously the associated transformations,

$$\alpha(\vec{p})\to e^{-i\chi^{+}(\vec{p})}\alpha(\vec{p})\,,\quad \beta(\vec{p})\to e^{-i\chi^{-}(\vec{p})}\beta(\vec{p})\,, \tag{137}$$

that preserves the form of ψ, we find that the operators $\tilde{P}_i$ keep their forms while the other generators are changing, e. g. the Hamiltonian operators transform as, $\tilde{H}^{(\pm)} \to \tilde{H}^{(\pm)} + p^i\partial_{p^i}\chi^{\pm}(\vec{p})$. Obviously, these transformations are nothing other than unitary transformations among equivalent UIRs. Note that thanks to this mechanism one can fix suitable phases for determining desired forms of the basis generators keeping thus under control the flat and rest limits of these operators in the Dirac [19] or scalar [16, 28] field theory on M.

At the level of QFT, the operators $\{\mathbf{X}_{(AB)}\}$, given by Eq. (124) where we introduce the differential operators (126) -(129), generate a reducible operator valued CR which can be decomposed as the orthogonal sum of CRs - generated by $\{\mathbf{X}^{(+)}_{(AB)}\}$ and $\{\mathbf{X}^{(-)}_{(AB)}\}$ - that are equivalent between themselves and equivalent to the UIRs $\left(\frac{1}{2}, \nu_{\pm}\right)$ of the $sp(2,2)$ algebra. These one-particle operators are the principal conserved quantities of the Dirac theory corresponding to the de Sitter isometries via Noether theorem.

It is remarkable that in our formalism we have $\tilde{X}^{(+)}_{AB} = \tilde{X}^{(-)}_{AB}$ which means that the particle and antiparticle sectors bring similar contributions such that we can say that these quantities are ***additive***, e. g., the energy of a many particle system is the sum of the individual energies of particles and antiparticles.

Other important conserved one-particle operators are the components of the Pauli - Lubanski operator,

$$\mathbf{W}_A = \mathbf{W}_A^{(+)} + \mathbf{W}_A^{(-)} = \int d^3p \left[\alpha^\dagger(\vec{p})\tilde{W}_A^{(+)}\alpha(\vec{p}) + \beta^\dagger(\vec{p})\tilde{W}_A^{(-)}\beta(\vec{p})\right] , \quad (138)$$

as given by Eqs. (131)-(133).

The Casimir operators of QFT have to be calculated according to Eqs. (92) and (99) but by using the one-particle operators $\mathbf{X}_{(AB)}$ and $\mathbf{W}_A$ instead of $\tilde{X}_{(AB)}$ and $\tilde{W}_A$. We obtain the following one-particle contributions

$$\mathbf{C}_1 = \left(m^2 + \frac{3}{2}\omega^2\right)\mathbf{N} + \cdots , \quad \mathbf{C}_2 = \frac{3}{4}\left(m^2 + \frac{1}{4}\omega^2\right)\mathbf{N} + \cdots , \quad (139)$$

where $\mathbf{N} = \mathbf{N}^{(+)} + \mathbf{N}^{(-)}$ is the usual operator of the total number of particles and antiparticles.

Thus the additivity holds for the entire theory of the spacetime symmetries in contrast with the conserved charges of the internal symmetries that take different values for particles and antiparticles as, for example, the charge operator corresponding to the $U(1)_{em}$ gauge symmetry [21] that reads $\mathbf{Q} = q(\mathbf{N}^{(+)} - \mathbf{N}^{(-)})$.

6. Concluding remarks

The principal conclusion is that the QFT on the de Sitter background has similar features as in the flat case. Thus the covariant quantum fields transforming according to CRs induced by the reps. of the group $\hat{G} = SL(2, \mathbb{C})$ that must be equivalent to orthogonal sums of UIRs of the group $S(M) = Sp(2, 2)$ whose specific invariants depend only on particle masses and spins.

The example is the spinor CR of the Dirac theory that is induced by the linear rep. $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$ of the group $\hat{G}$ but is equivalent to the orthogonal sum of two equivalent UIRs of the group $S(M)$ labelled by $(\frac{1}{2}, \nu_{\pm})$.

Thus at least in the case of the Dirac field we recover a similar conjuncture as in the Wigner theory of the induced reps. of the Poincaré group in special relativity.

However, the principal difference is that the transformations of the Wigner UIRs can be written in closed forms while in our case this cannot be done because of the technical difficulties in solving the integrals (122) and (123). For this reason we were forced to restrict ourselves to study only the reps. of the corresponding algebras.

This is not an impediment since physically speaking we are interested to know the properties of the basis generators (in configurations or momentum rep.) since these give rise to the conserved observables (i. e. the one-particle operators) of QFT, associated to the de Sitter isometries.

It is remarkable that the particle and antiparticle sectors of these operators bring additive contributions since the particle and antiparticle operators transform alike under isometries just as in special relativity.

Notice that this result was obtained by Nachtmann [16] for the scalar UIRs but this is less relevant as long as the generators of the scalar rep. depend only on m^2. Now we see that the generators of the spinor rep. which have spin terms depending on m preserve this property such that we can conclude that all the one-particle operators corresponding to the de Sitter isometries are additive, regardless the spin.

The principal problem that remains unsolved here is how to build on the de Sitter manifolds a Wigner type theory able to define the structure of the covariant fields without using field equations.

Appendix

A: Finite-dimensional reps. of the $sl(2,\mathbb{C})$ algebra

The standard basis of the $sl(2,\mathbb{C})$ algebra is formed by the generators $\vec{J} = (J_1, J_2, J_3)$ and $\vec{K} = (K_1, K_2, K_3)$ that satisfy [30, 8]

$$[J_i, J_j] = i\varepsilon_{ijk}J_k\,, \quad [J_i, K_j] = i\varepsilon_{ijk}K_k\,, \quad [K_i, K_j] = -i\varepsilon_{ijk}J_k\,, \tag{140}$$

having the Casimir operators $c_1 = i\vec{J}\cdot\vec{K}$ and $c_2 = \vec{J}^2 - \vec{K}^2$. The linear combinations $A_i = \frac{1}{2}(J_i + iK_i)$ and $B_i = \frac{1}{2}(J_i - iK_i)$ form two independent $su(2)$ algebras satisfying

$$[A_i, A_j] = i\varepsilon_{ijk}A_k\,, \quad [B_i, B_j] = i\varepsilon_{ijk}B_k\,, \quad [A_i, B_j] = 0\,. \tag{141}$$

Consequently, any finite-dimensional irreducible rep. (IR) $\tau = (j_1, j_2)$ is carried by the space of the direct product $(j_1)\otimes(j_2)$ of the UIRs (j_1) and (j_2) of the $su(2)$ algebras (A_i) and respectively (B_i). These IRs are labeled either by the $su(2)$ labels (j_1, j_2) or giving the values of the Casimir operators $c_1 = j_1(j_1+1) - j_2(j_2+1)$ and $c_2 = 2[j_1(j_1+1) + j_2(j_2+1)]$.

The fundamental reps. defining the $sl(2,\mathbb{C})$ algebra are either the IR $(\frac{1}{2}, 0)$ generated by $\{\frac{1}{2}\sigma_i, -\frac{i}{2}\sigma_i\}$ or the IR $(0, \frac{1}{2})$ whose generators are $\{\frac{1}{2}\sigma_i, \frac{i}{2}\sigma_i\}$. Their direct sum form the spinor

IR $\rho_s = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ of the Dirac theory. In applications it is convenient to consider ρ_s as the fundamental rep. since here invariant forms can be defined using the Dirac conjugation.

The spin basis of the IR τ can be constructed as the direct product,

$$|\tau, s\sigma\rangle = \sum_{\lambda_1+\lambda_2=\sigma} C^{s\sigma}_{j_1\lambda_1, j_2\lambda_2} |j_1, \lambda_1\rangle \otimes |j_2, \lambda_2\rangle \,, \tag{142}$$

of $su(2)$ canonical bases where the Clebsh-Gordan coefficients [8] give the spin content of the IR τ, i. e. $s = j_1 + j_2, j_1 + j_2 - 1, ..., |j_1 - j_2|$. Note that for integer values of spin we can resort to the tensor bases constructed as direct products of the vector bases $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ of the vector IR $(j = 1)$ (which satisfy $|1, \pm 1\rangle = \frac{1}{\sqrt{2}}(\vec{e}_1 \pm i\vec{e}_2)$ and $|1, 0\rangle = \vec{e}_3$).

Given the IR $\tau = (j_1, j_2)$ we say that its adjoint IR is $\dot{\tau} = (j_2, j_1)$ and observe that these have the same spin content while their generators are related as $\vec{J}^{(\dot{\tau})} = \vec{J}^{(\tau)}$ and $\vec{K}^{(\dot{\tau})} = -\vec{K}^{(\tau)}$. On the other hand, the operators $\vec{A}$ and $\vec{B}$ are Hermitian since we use UIRs of the $su(2)$ algebra. Consequently, we have $\vec{J}^{+} = \vec{J}$ and $\vec{K}^{+} = -\vec{K}$ for any finite-dimensional IR of the $sl(2, \mathbb{C})$ algebra, such that we can write

$$(\vec{J}^{(\tau)})^{+} = \vec{J}^{(\tau)} \,, \quad (\vec{K}^{(\tau)})^{+} = \vec{K}^{(\dot{\tau})} \,. \tag{143}$$

Hereby we conclude that invariant forms can be constructed only when we use reducible reps. $\rho = \cdots \tau_1 \oplus \tau_2 \cdots \dot{\tau}_1 \oplus \dot{\tau}_2 \cdots$ containing only pairs of adjoint reps.. Then the matrix $\gamma_{(\rho)}$ may be constructed with the matrix elements

$$\langle \tau_1, s_1\sigma_1 | \gamma_{(\rho)} | \tau_2, s_2\sigma_2 \rangle = \delta_{\tau_1 \dot{\tau}_2} \delta_{s_1 s_2} \delta_{\sigma_1 \sigma_2} \,. \tag{144}$$

Note that the canonical basis $\{\tau, j\lambda\rangle\}$ defines the ***chiral*** rep. while a new basis in which $\gamma_{(\rho)}$ becomes diagonal gives the so called standard rep.. This terminology comes from the Dirac theory where $\gamma_{(\rho_s)} = \gamma^0$ is the Dirac matrix that may have these reps. [27].

B: Some properties of Hankel functions

According to the general properties of the Hankel functions [31], we deduce that those used here, $H^{(1,2)}_{\nu_\pm}(z)$, with $\nu_\pm = \frac{1}{2} \pm i\mu$ and $z \in \mathbb{R}$, are related among themselves through $[H^{(1,2)}_{\nu_\pm}(z)]^* = H^{(2,1)}_{\nu_\mp}(z)$ and satisfy the identities

$$e^{\pm \pi k} H^{(1)}_{\nu_\mp}(z) H^{(2)}_{\nu_\pm}(z) + e^{\mp \pi k} H^{(1)}_{\nu_\pm}(z) H^{(2)}_{\nu_\mp}(z) = \frac{4}{\pi z} \,. \tag{145}$$

References

[1] R. M. Wald, ***General Relativity***, (Univ. of Chicago Press, Chicago and London, 1984)

[2] H. B. Lawson Jr. and M.-L. Michaelson, ***Spin Geometry*** (Princeton Univ. Press. Princeton, 1989).

[3] I. I. Cotăescu, ***J. Phys. A: Math. Gen.*** **33**, 9177 (2000).

[4] I. I. Cotăescu, ***Europhys. Lett.*** **86**, 20003 (2009).

[5] I. I. Cotăescu, ***Mod. Phys. Lett. A*** **28** (2013) 13500033-1

[6] E. Wigner, ***Ann. Math.*** **40** 149 (1939).

[7] G. Mackey, ***Ann. Math.*** **44**, 101 (1942).

[8] W.-K. Tung, ***Group Theory in Physics*** (World Sci., Philadelphia, 1984).

[9] J. Dixmier, ***Bull. Soc. Math. France*** **89**, 9 (1961).

[10] B. Takahashi, ***Bull. Soc. Math. France*** **91**, 289 (1963).

[11] B. Allen and T. Jacobson, ***Commun. Math. Phys.*** **103**, 669 (1986).

[12] N. C. Tsamis and R. P. Woodard, ***J. Math. Phys.*** **48**, 052306 (2007).

[13] J.-P. Gazeau and M.V. Takook, ***J. Math. Phys.*** **41**, 5920 (2000) 5920. Gaz

[14] P. Bartesaghi, J.-P. Gazeau, U. Moschella and M. V. Takook, *Class. Quantum. Grav.* **18** (2001) 4373.

[15] T. Garidi, J.-P. Gazeau and M. Takook, *J.Math.Phys.* **44**, 3838 (2003).

[16] O. Nachtmann, *Commun. Math. Phys.* **6**, 1 (1967).

[17] I. I. Cotăescu *Mod. Phys. Lett. A* **28**, 1350033 (2013).

[18] I. I. Cotăescu, *Phys. Rev. D* **65**, 084008 (2002).

[19] I. I. Cotăescu, *Mod. Phys. Lett. A* **26**, 1613 (2011).

[20] I. I. Cotăescu, *GRG* **43**, 1639 (2011).

[21] I. I. Cotăescu and C. Crucean, *Phys. Rev. D* **87**, 044016 (2013).

[22] I. I. Cotaĕscu and D.-M. Băltăţeanu, *Mod. Phis. Lett. A* **30**, 1550208 (2014).

[23] N. D. Birrel and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge 1982).

[24] S. Weinberg, *Phys. Rev.* 133, B1318 (1964)

[25] B. Carter and R. G. McLenaghan, *Phys. Rev. D* **19**, 1093 (1979).

[26] S. Drell and J. D. Bjorken, *Relativistic Quantum Fields* (Me Graw-Hill Book Co., New York 1965).

[27] B. Thaller, *The Dirac Equation* (Springer Verlag, Berlin Heidelberg, 1992).

[28] I. I. Cotăescu and G. Pascu, ***Mod. Phys. Lett. A*** **28**, 1350160 (2013).

[29] G. Börner and H. P. Dürr, ***Nuovo Cimenta*** **LXIV A**, 669 (1969).

[30] M. A. Naimark, ***Linear reps. of the Lorentz Group*** (Pergamon Press, Oxford, 1964).

[31] I. S. Gradshtein and I. M. Ryzhik, ***Table of Integrals, Series and Products*** (Academic Press Inc., San Diego 1980).