Lecture 3 - Applications

Quintessence and K-essence
DE/DM unification
Tachyon condensate in a Braneworld

Quintessence

The so called coincidence problem is somewhat ameliorated in ***quintessence*** – a canonical scalar field with selfinteraction effectively providing a model for dark-energy and accelerating expansion (comparable to slow roll inflation) today

P.Ratra, J. Peebles PRD 37 (1988)

$$S = \int d^4x \, \mathcal{L}(X, \varphi)$$

$$X = g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu}$$

$$\mathcal{L} = \frac{1}{2} X - V(\varphi)$$

$$T_{\mu\nu} = \frac{2}{\sqrt{-\det g}} \frac{\delta S}{\delta g^{\mu\nu}} = \varphi_{,\mu} \varphi_{,\nu} - \mathcal{L} g_{\mu\nu}$$

Field theory description of a perfect fluid if $X>0$

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu - p g_{\mu\nu}$$

where

$$u_\mu = \frac{\varphi_{,\mu}}{\sqrt{X}}$$

$$p = \mathcal{L} = \frac{1}{2}X - V(\varphi) \qquad \rho = X - \mathcal{L} = \frac{1}{2}X + V(\varphi)$$

A suitable choice of *V(φ)* yields a desired cosmology, or vice versa: from a desired equation of state $p=p(\rho)$ one can derive the Lagrangian of the corresponding scalar field theory

Phantom quintessence

Phantom energy is a substance with negative pressure such that $|p|$ exceeds the energy density so that the null energy condition (NEC) is violated, i.e., $p+\rho<0$. Phantom quintessence is a scalar field with a negative kinetic term

$$p = \mathcal{L} = -\frac{1}{2}X - V(\varphi) \qquad \rho = -\frac{1}{2}X + V(\varphi)$$

Obviously, for $X>0$ we have $p+\rho<0$ which demonstrates a violation of NEC! This model predicts a catastrophic end of the Universe, the so-called Big Rip - the total collapse of all bound systems.

k-essence

k-essence is a generalized quintessence which was first introduced as a model for inflation . A minimally coupled k-essence model is described by

$$S = \int d^4x \sqrt{-g}\left[-\frac{R}{16\pi G} + \mathcal{L}(\varphi, X)\right]$$

where $\mathcal{L}$ is the most general Lagrangian, which depends on a single scalar field of dimension of length , and on the dimensionless quantity $X = g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}$ For X>0 , the energy momentum tensor takes the perfect fluid form

$$T_{\mu\nu} = 2\mathcal{L}_X\, \varphi_{,\mu}\varphi_{,\nu} - \mathcal{L} g_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}$$

where $\mathcal{L}_X = \dfrac{\partial \mathcal{L}}{\partial X}$

The associated hydrodynamic quantities are

$$p = \mathcal{L}(\varphi, X) \qquad \rho = 2X\mathcal{L}_X(\varphi, X) - \mathcal{L}(\varphi, X).$$

Examples

Kinetic k-essence The Lagrangian is a function of *X* only. In this case

$$p = \mathcal{L}(X) \qquad \rho = 2X\mathcal{L}_X(X) - \mathcal{L}(X)$$

To this class belong the ghost condensate

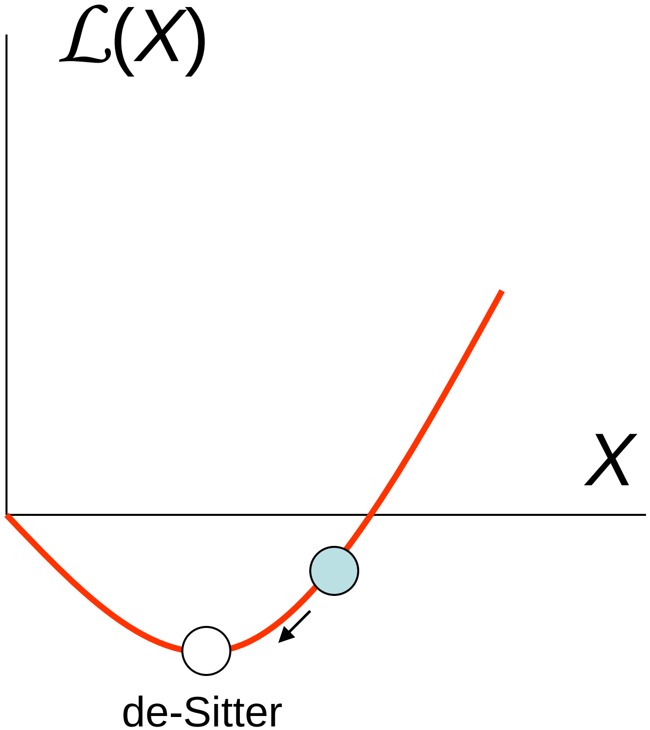
$$\mathcal{L}(X) = A(1 - X)^2 + B$$

and the scalar Born-Infeld model

$$\mathcal{L}(X) = -A\sqrt{1 - X}$$

Ghost condensate model

N. Arkani-Hamed et al , JHEP **05** (2004)
R.J. Scherrer, PRL **93** (2004)



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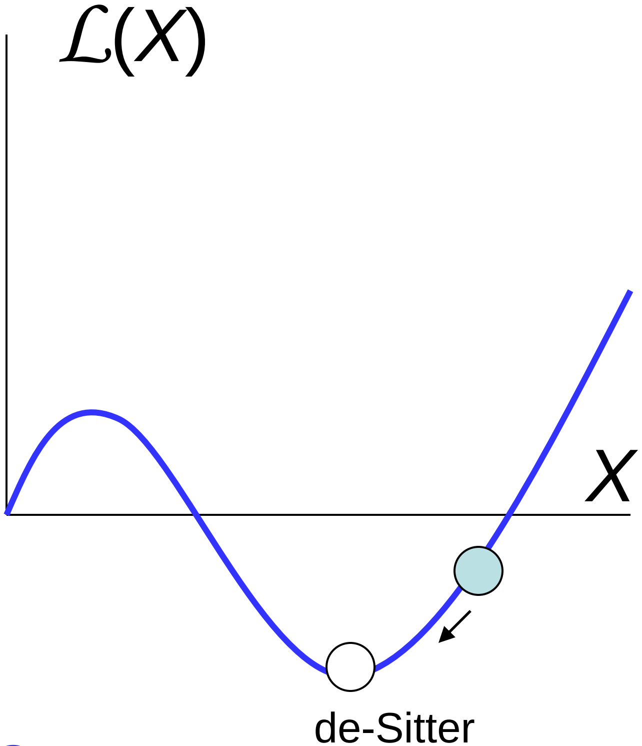


Slow roll towards
the minimum

Exercise No 14: For a general kinetic k-essence, i.e. when $\mathcal{L} = \mathcal{L}(X)$ using Hamilton equations show that $\pi(a) = C / a^3$, where C is a constant

Exercise No 15: For a ghost condensate model with Lagrangian $\mathcal{L} = B^2 (X^2 - 1)$

a) express X, $\mathcal{L}$ and $\mathcal{H}$ in terms of π

b) find a as a function of t ,

c) using Hamilton equations find φ as a function of a

DE/DM Unification – Quartessence

The astrophysical and cosmological observational data can be accommodated by combining baryons with conventional CDM and a simple cosmological constant providing DE. This *Λ*CDM model, however, faces the fine tuning and coincidence problems associated to *Λ*.

Another interpretation of this data is that DM/DE are different manifestations of a common structure. The general class of models, in which a unification of DM and DE is achieved through a single entity, is often referred to as *quartessence.* Most of the unification scenarios that have recently been suggested are based on *k-essence* type of models including Ghost Condensate, various variants of the Chaplygin Gas

Chaplygin Gas

An exotic fluid with an equation of state

$$p = -\frac{A}{\rho}$$

The first definite model for a dark matter/energy unification

A. Kamenshchik, U. Moschella, V. Pasquier, PLB **511** (2001)
N.B., G.B. Tupper, R.D. Viollier, PLB **535** (2002)
J.C. Fabris, S.V.B. Goncalves, P.E. de Souza, GRG **34** (2002)

The Chaplygin gas model is equivalent to (scalar) Born-Infeld description of a D-brane:

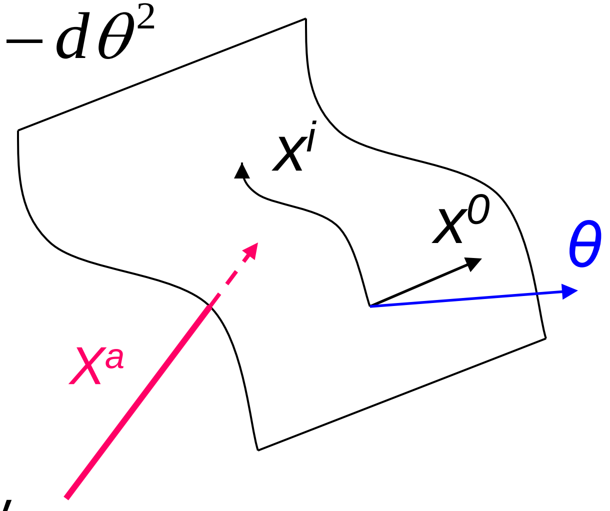
Consider a 3-brane in 4+1 dim. Bulk. We can choose the bulk coordinates such that $X^{\mu} = x^{\mu}$, $\mu = 0,..3$, and let the fifth coordinate $X^4 \equiv \theta$ be normal to the brane. In the simplest case (no warp factor dependent on the 5th coordinate) the bulk line element is

$$ds^2_{(5)} = G_{ab}(X)dX^a dX^b = g_{\mu\nu}(x)dx^{\mu}dx^{\nu} - d\theta^2$$

so

$$G_{\mu\nu} = g_{\mu\nu} \quad \text{for} \quad \mu = 0...3$$

$$G_{\mu 4} = 0 \qquad G_{44} = -1$$



This yields the Dirac-Born-Infeld type of theory

$$S_{\mathrm{DBI}} = -\sigma \int dx^4 \sqrt{1 - X} \qquad \text{where} \qquad X = g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}$$

Exercise No 16: Derive S_{DBI} (see Exerc. No 12)

Scalar Born-Infeld theory

$$\mathcal{L}_{\text{DBI}} = -\sqrt{A}\sqrt{1-X} \qquad X = g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}$$

$$T_{\mu\nu} = 2\frac{\partial \mathcal{L}_{\text{BI}}}{\partial X}\varphi_{,\mu}\varphi_{,\nu} - \mathcal{L}_{\text{BI}}\, g_{\mu\nu}$$

perfect fluid

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}$$ with $$u_\mu = \frac{\varphi_{,\mu}}{\sqrt{X}}$$

Identifying

$$p = -\sigma\sqrt{1-X} \qquad \sigma \equiv \sqrt{A}$$

$$\rho = 2X\frac{\partial p}{\partial X} - p = \frac{\sigma}{\sqrt{1-X}}$$

We obtain the Chaplygin gas equation of state

$$p = -\frac{\sigma^2}{\rho}$$

In a homogeneous model the conservation equation yields the Chaplygin gas density as a function of the scale factor a

$$\rho(a) = \sigma\sqrt{1 + b/a^6} \qquad (37)$$

where b is an integration constant.

The Chaplygin gas thus interpolates between dust ($\rho \sim a^{-3}$) at high redshifts and a cosmological constant ($\rho \sim \sigma$) today and hence yields a correct homogeneous cosmology

Exercise No 17: Derive (37) using energy-momentum conservation

Exercise No 18: For the scalar Born-Infeld theory with $\mathcal{L} = -\sigma(1-X)^{1/2}$

a) express X, $\mathcal{L}$ and $\mathcal{H}$ in terms of π

b) Find t as a function of a, and show that a grows exponentially for large t

c) using Hamilton equations find φ as a function of a in terms of the Gauss hypergeometric function ${}_2F_1$

d) Using properties of ${}_2F_1$ show the following asymptotic behavior

$$\phi \propto a^{-3} \quad \text{for } a \to \infty$$

$$\phi \propto a^{3/2} \quad \text{for } a \to 0$$

Problems with Nonvanishing Sound Speed

To be able to claim that a field theoretical model actually achieves unification, one must be assured that initial perturbations can evolve into a deeply nonlinear regime to form a gravitational condensate of superparticles that can play the role of CDM. The inhomogeneous Chaplygin gas based on a Zel'dovich type approximation has been proposed,

N.B., G.B. Tupper, R.D. Viollier, PLB **535** (2002)

and the picture has emerged that on caustics, where the density is high, the fluid behaves as cold dark matter, whereas in voids, $w=p/\rho$ is driven to the lower bound -1 producing

acceleration as dark energy. In fact, for this issue, the usual Zel’dovich approximation has the shortcoming that the effects of finite sound speed are neglected.

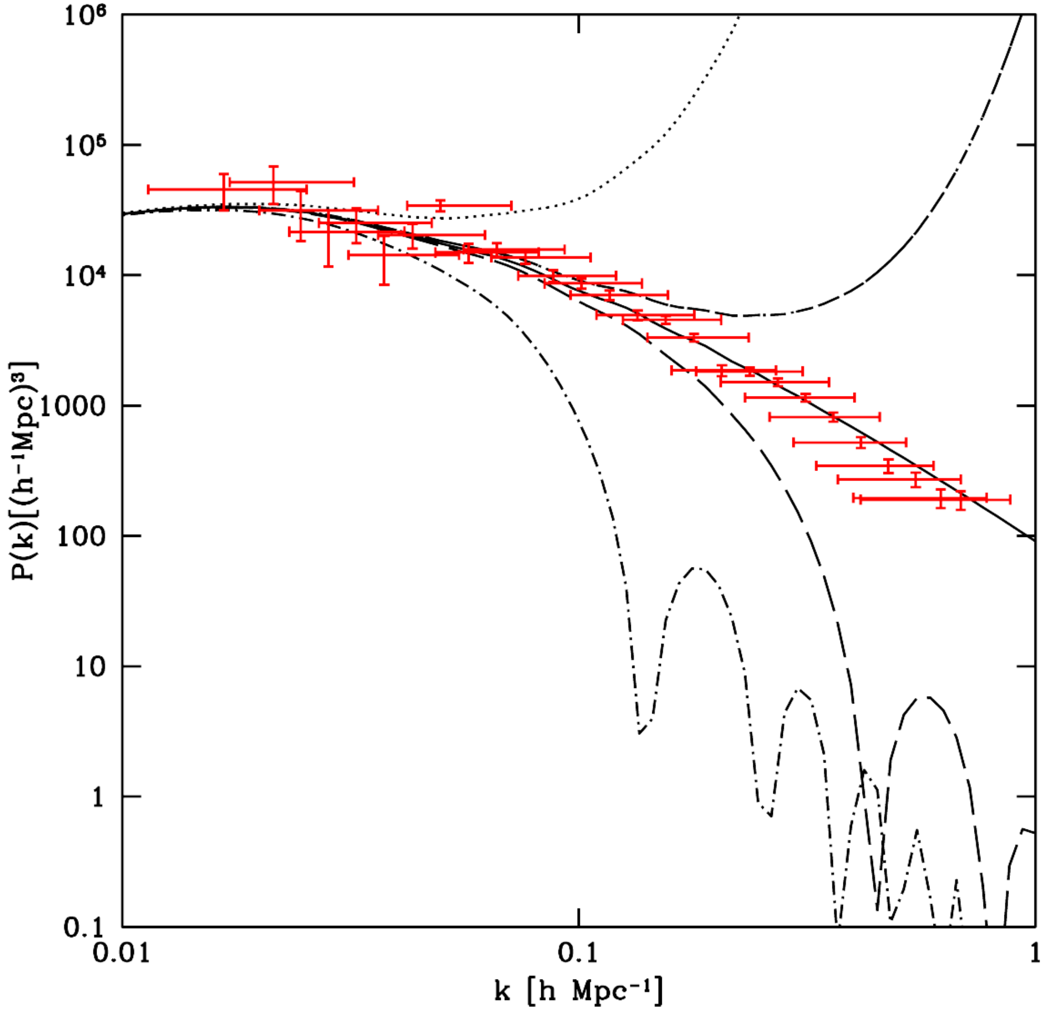
In fact, all models that unify DM and DE face the problem of sound speed related to the well-known Jeans instability. A fluid with a nonzero sound speed has a characteristic scale below which the pressure effectively opposes gravity. Hence the perturbations of the scale smaller than the sonic horizon will be prevented from growing.

- Soon after the Chaplygin gas was proposed as a model of unification it has been shown that the naive model does not reproduce the mass power spectrum

H.B. Sandvik et al PRD **69** (2004)

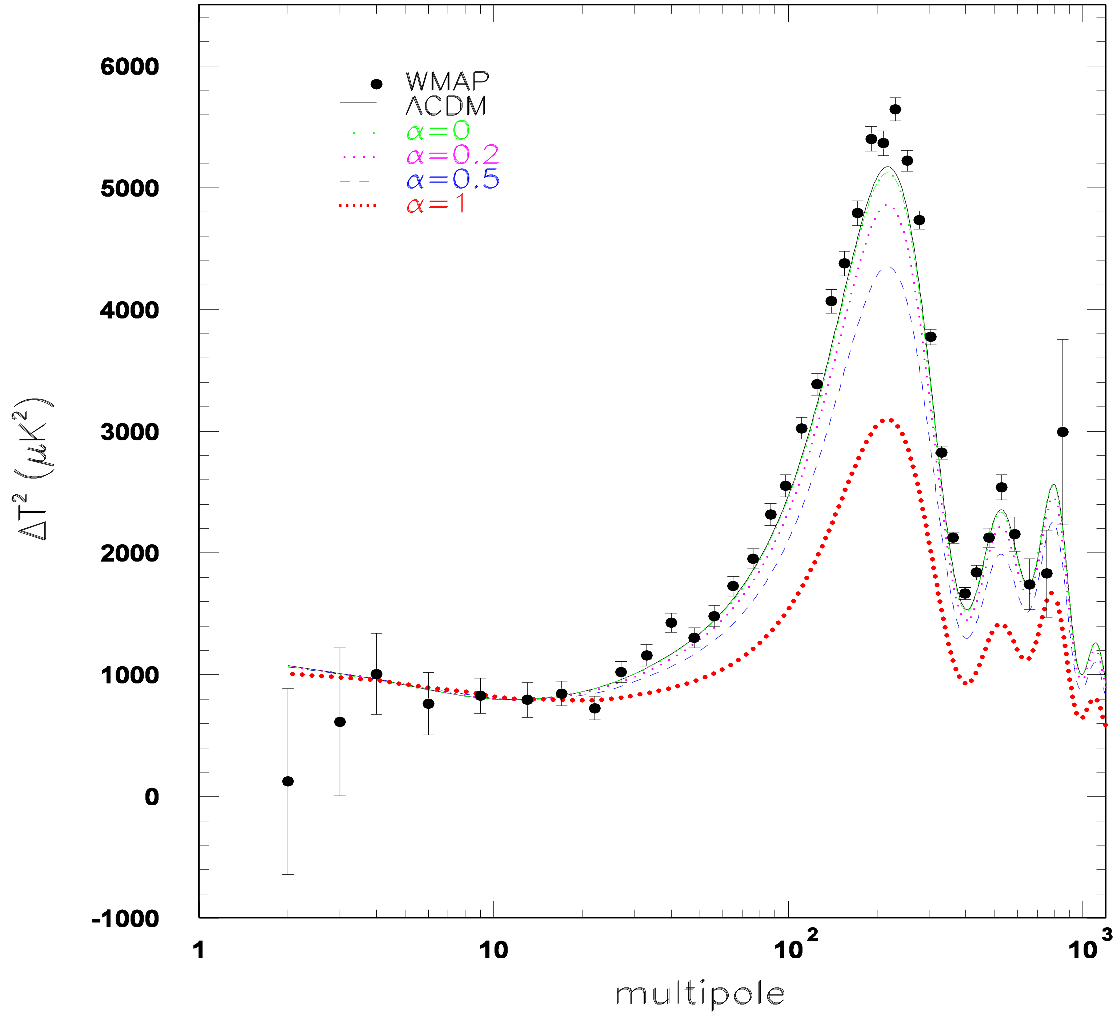
and the CMB

D. Carturan and F. Finelli, PRD **68** (2003);
L. Amendola et JCAP **07** (2003)



Power spectrum for $p = -A/\rho^{\alpha}$ for various α

H.B. Sandvik et al, PRD 2004



CMB spectrum for $p=-A/\rho^{\alpha}$ for various α

L. Amendola et al, JCAP 2003

The physical reason is a nonvanishing sound speed. Although the adiabatic speed of sound

$$c_s^2 = \frac{\partial p}{\partial \rho}\bigg|_S = \frac{A}{\rho^2}$$

is small until $a \sim 1$, the accumulated comoving size of the acoustic horizon

$$d_s = \int dt\,\frac{c_s}{a} \simeq H_0^{-1} a^{7/2}$$

reaches Mpc scales by redshifts of about $z \sim 20$, thus frustrating the structure formation at galactic and subgalactic scales. This may be easily demonstrated in a simple spherical model.

N.B., R. Lindebaum, G.B. Tupper, R.D. Viollier, JCAP **0411** (2003)

Generalized Chaplygin Gas

Another model was proposed in an attempt to solve the structure formation problem and has gained a wide popularity. The generalized Chaplygin gas is defined as

$$p = -\frac{A}{\rho^{\alpha}}, \qquad 0 \le \alpha \le 1$$

The additional parameter does afford greater flexibility: e.g. for small α the sound horizon $d_s \sim \sqrt{\alpha} a^2 / H_0$ and thus by fine tuning $\alpha < 10^{-5}$, the data can be perturbatively accommodated

M.C. Bento, O. Bertolami, and A.A. Sen, PRD **66** (2002)

Other modifications

- The generalized Chaplygin gas in a modified gravity approach, reminiscent of Cardassian models

$$H^2 = \frac{8\pi G}{3}\left(A + \rho^{1+\alpha}\right)^{1/(1+\alpha)}$$

T. Barreiro and A.A. Sen, PRD **70** (2004)

- A deformation of the Chaplygin gas – Milne-Born-Infeld theory

$$\mathcal{L} = -\sqrt{A}\sqrt{X - bX^2}$$

M. Novello, M. Makler, L.S. Werneck and C.A. Romero, PRD **71** (2005)

- Variable Chaplygin gas $p \sim -a^n / \rho$

Zong-Kuan Guo, Yuan-Zhong Zhang, astro-ph/0506091, PLB (2007)

Tachyon Condensate

The failure of the simple Chaplygin gas (CG) does not exhaust all the possibilities for quartessence. The Born-Infeld Lagrangian is a special case of the string-theory inspired tachyon Lagrangian in which the constant *A* is replaced by a potential

$$\mathcal{L} = -V(\varphi)\sqrt{1 - g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}}\ .$$

Tachyon models are a particular case of k-essence. It was noted in that (in the FRW cosmology), the tachyon model is described by the CG equation of state in which the constant *A* is replaced by a function of the cosmological scale factor $p \sim -a^n/\rho$ so the model was dubbed “variable Chaplygin gas”.

Zong-Kuan Guo et al astro-ph/0506091, PLB (2007)

A preliminary analysis of a unifying model based on the tachyon type Lagrangian has been carried out in

N.B., G.B. Tupper, R.D. Viollier, PRD **80** (2009)

for a potential of the form

$$V(\varphi) = V_n \varphi^{2n} \qquad n = 0, 1, 2$$

n=0 gives the Dirac-Born-Infeld description of a D-brane - equivalent to the Chaplygin gas

It may be shown that the model with ***n≠0*** effectively behaves as a ***variable*** Chaplygin gas, with $p \sim -a^{6n}/\rho$. The much smaller sonic horizon $d_s \sim a^{(7/2+3n)}/H_0$ enhances condensate formation by 2 orders of magnitude over the simple Chaplygin gas. Hence this type of model may salvage the quartessence scenario.

Cosmological evolution of the tachyon condensate

In Lecture 2 we have derived a tachyon Lagrangian of the form

$$\mathcal{L} = -V(\theta)\sqrt{1 - g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}}$$

in the context of a dynamical brane moving in a 4+1 background with a general warp

$$ds^2_{(5)} = \psi^2(y)g^{\mu\nu}dx^\mu dx^\nu - dy^2 = \frac{1}{\chi^2(z)}(g^{\mu\nu}dx^\mu dx^\nu - dz^2)$$

where $\chi(z) = \dfrac{1}{\psi(y)}$ $\qquad z = \int dz / \psi(y)$

The field θ is identified with the 5-th coordinate z and the potential is related to the warp

$$V(\theta) = \sigma / \chi^4(\theta)$$

Naw we assume our Braneworld to be a spatially flat FRW universe with line element

$$ds^2 = g^{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t)\left[dr^2 + r^2 d\Omega^2\right]$$

$a(t)$ – cosmological scale

The Lagrangian $\mathcal{L}$ is then

$$\mathcal{L} = -V(\theta)\sqrt{1-\dot{\theta}^2} = -V(\theta)\sqrt{1-X}$$

The Hamiltonian corresponding to $\mathcal{L}$ is easily derived and is very simple

$$\mathcal{H} = \frac{V}{\sqrt{1-X}} == \sqrt{V^2+\pi^2} \qquad (38)$$

where the conjugate momentum π is related to θ and its time derivative via

$$\pi = \frac{V(\theta)\dot{\theta}}{\sqrt{1-\dot{\theta}^2}}$$

The Hamilton equations derived previously

$$\dot{\theta} = \frac{\partial \mathcal{H}}{\partial \pi} \qquad \dot{\pi} + 3H\pi = -\frac{\partial \mathcal{H}}{\partial \theta} \tag{39}$$

In this case become

$$\dot{\theta} = \frac{\pi}{\mathcal{H}} \qquad \dot{\pi} + 3H\pi + \frac{VV'}{\sqrt{V^2 + \pi^2}} = 0 \tag{40}$$

Where H is the Hubble constant in BW cosmology

$$H = \sqrt{\frac{8\pi G}{3}\mathcal{H}\left(1 + \frac{2\pi G}{3k^2}\mathcal{H}\right)} \tag{41}$$

Of special physical importance is the equation of state

$$w = \frac{p}{\rho} = -(1 - X) = -\frac{1}{1 + \pi^2 / V^2}; \qquad X \equiv \dot{\theta}^2$$

Exercise No 19: Derive (40) using (38) and (39)

Consider as an example the tachyon potential of the form

$$V = \sigma(k\theta)^{-n}$$

The model can be solved analytically for specific powers *n* in two cases:

1) In the low energy regime (relevant for today's cosmology) the analytical solution may be found for $n=2$

L.R. Abramo and F. Finelli, PLB 575 (2003)

2)In the high energy regime (relevant during the slow roll period of inflation) the analytical solution may be found for $n=1$

It is convenient to put the equations in dimensionless form. We rescale all dimensionful variables as

$$t = \tau / k, \quad H = kh, \quad \theta \to \theta / k, \quad \pi \to \sigma\pi$$

$$\mathcal{H} = \sigma\rho, \quad \mathcal{L} = \sigma p$$

The potential becomes simply $V = \theta^{-n}$ and Hamilton equations read

$$\dot{\theta} = \frac{\theta^n \pi}{\sqrt{1 + \pi^2 \theta^{2n}}}$$

$$\dot{\pi} = -3h\pi + \frac{n}{\theta^{n+1}\sqrt{1 + \pi^2 \theta^{2n}}}$$

with

$$h = \sqrt{\frac{\kappa^2}{3}\rho\left(1 + \frac{\kappa^2}{12}\rho\right)}$$

where we have introduced a dimensionless coupling constant

$$\kappa^2 = \frac{8\pi G_{\mathrm{N}}}{k^2}\sigma$$

We will try to solve Hamilton's equations by an ansatz:

$$\pi(\theta) = c\theta^{m-n}$$

from which we find

$$\dot{\pi} = c(m-n)\theta^{m-n-1}\dot{\theta}$$

This, together with the first Hamilton equation

$$\dot{\theta} = \frac{c\theta^m}{\sqrt{1 + c^2\theta^{2m}}}$$

yields

$$\dot{\pi} = c^2(m-n)\theta^{2m-n-1}\left(1 + c^2\theta^{2m}\right)^{-1/2}$$

We will seek a solution assuming three scenarios
a)m=0, b)m>0 and c)m<0

The physicall meaning of these three scenarios can be seen by looking at the equation of state

$$w = \frac{p}{\rho} = -(1 - X) = -\frac{1}{(1 + c^2\theta^{2m})}$$

a)m=0 $\Longrightarrow$ w= -const – some form of dark energy

a)m>0 $\Longrightarrow$ for large θ $w \to 0$ – some form of pressureless matter or dust

a)m<0 $\Longrightarrow$ for large θ $w \to -1$ – de Sitter dark energy (cosmological constant like)

We will distingush two regimes:

1)The low energy regime (relevant for today's cosmology)
In this case we have $G\mathcal{H}/k^2 \ll 1$ (or $\kappa^2\rho \ll 1$) so we can neglect the quadratic correction in the first Friedmann equation

Then the second Hamilton equation yields an identity

$$c^2(m-n)\theta^{2m-n-1} = n\theta^{-n-1} - c\sqrt{3}\kappa\theta^{m-3n/2}(1+c^2\theta^{2m})^{3/4} \qquad (42a)$$

2)The high energy regime (relevant for the early cosmology)

In this case we have $G\mathcal{H}/k^2 \gg 1$ (or $\kappa^2\rho \gg 1$) so the quadratic correction in the first Friedmann equation dominates and we find another identity

$$c^2(m-n)\theta^{2m-n-1} = n\theta^{-n-1} - \frac{1}{2}c\kappa^2\theta^{m-2n}(1+c^2\theta^{2m}) \qquad (42b)$$

Exercise No 20: Derive (42a,b) using Hamilton's equations

1)The low energy regime

a)*m=0*

There exist a solution to (42b) provided n=2 in which case we find

$$c = \frac{2\sqrt{2}}{3\kappa^2}\left(1+\sqrt{1+\frac{9}{4}\kappa^4}\right)^{1/2} \qquad (43)$$

And we obtain

$$a = a_0 (kt)^p \qquad \text{where} \qquad p = \frac{1}{3}\left(1+\sqrt{1+\frac{9}{4}\kappa^4}\right) \qquad (44)$$

$$\theta = \frac{c}{\sqrt{1+c^2}}t + \text{const} \qquad \rho = \frac{1+c^2}{c}\left(\frac{a}{a_0}\right)^{-1/p} \qquad (45)$$

Exercise No 21: Derive (43), (44) and (45)

a) *m>0*

The analytic solution cannot be found. However, the identity (42) can be satisfied in the asymptotic regime of large θ. Eq. (42) simplifies

$$c^2(m-n)\theta^{2m-n-1}+c^{5/2}\sqrt{3}\kappa\theta^{(5m-3n)/2}=0 \qquad (46)$$

The solution exists provided $m=n-2$ so we must have $n>2$ in which case we find

$$c=\frac{4}{3\kappa^2} \qquad \theta=t+\text{const} \qquad (47)$$

$$a=a_0(kt)^{2/3} \qquad (48)$$

$$\rho=\rho_0(a)^{-3} \qquad (49)$$

This behavior is typical of dust!

Exercise No 22: Derive (46), (47), (48) and (49)

a) *m*<0

The analytic solution cannot be found. Again, the identity (42) can be satisfied in the asymptotic regime of large θ. In this case Eq. (42) becomes

$$-n\theta^{-n-1} + c\sqrt{3}\kappa\theta^{(2m-3n)/2} = 0 \quad (50)$$

The solution exists provided $m=n/2-1$ so we must have $n<2$ in which case we find

$$\theta = \theta_0 (kt)^{2/(4-n)} \quad (51)$$

$$a = a_0 \exp(kt)^{(4-2n)/(4-n)} \quad (52)$$

$$\rho = \rho_0 \left(\ln a / a_0\right)^{-2n/(4-2n)} \quad (53)$$

This is a “quasi de Sitter” and becomes de Sitter for n=0

L.R. Abramo and F. Finelli, PLB 575 (2003)

Exercise No 23: Derive (50)-(53)

2)The high energy regime

a)$m=0$

There exist a solution to (42b) provided $n=1$ in which case we find

$$c = \frac{2}{\kappa^2} \qquad (54)$$

And we obtain

$$a = a_0 (kt)^p \quad \text{where} \quad p = \frac{1}{3}\left(1 + \frac{\kappa^4}{4}\right) \qquad (55)$$

$$\theta = \frac{c}{\sqrt{1+c^2}} t + \text{const} \qquad \rho = \frac{1+c^2}{c}\left(\frac{a}{a_0}\right)^{-1/p} \qquad (56)$$

Exercise No 24: Derive (54), (55) and (56)

Exercise No 25: Derive asymptotic behavior (large θ) in the high energy regime for m>0 and m<0 from (42b) following the procedure outlined for the low energy regime,

Thank You

Thomas-Fermi correspondence

Under reasonable assumptions in the cosmological context there exist an equivalence

Complex scalar field theories (canonical or phantom)	⟷	Kinetic k-essence type of models

Consider

$$\mathcal{L} = g^{\mu\nu}\Phi^*{}_{,\mu}\Phi_{,\nu} - V(|\Phi|^2/m^2) \qquad \Phi = \frac{\phi}{\sqrt{2}}e^{-im\theta}$$

Thomas-Fermi approximation $\phi_{,\mu} \ll m\phi$

$$\Rightarrow \quad \Phi_{,\mu} = -i\Phi\theta_{,\mu}$$

$\Rightarrow$ TF Lagrangian $\mathcal{L}_{\mathrm{TF}}/m^4 = XY - U(Y)$

where $X = g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}$ $\quad Y = \dfrac{\phi^2}{2m^2}$ $\quad U(Y) = V(Y)/m^4$

Equations of motion for φ and θ

$$X - \frac{\partial U}{\partial Y} = 0 \qquad\qquad (Yg^{\mu\nu}\theta_{,\nu})_{;\mu} = 0$$

We now define the potential $W(X)$ through a Legendre transformation

$$W(X) + U(Y) = XY \qquad\qquad U_Y \equiv \frac{\partial U}{\partial Y}$$

with $\boldsymbol{X = U_Y}$ and $\boldsymbol{Y = W_X}$

$$W_X \equiv \frac{\partial W}{\partial X}$$

correspondence

Complex scalar FT

$$\mathcal{L} = g^{\mu\nu}\Phi^*_{,\mu}\Phi_{,\nu} - V(|\Phi|^2/m^2)$$

$$\Phi = \frac{\phi}{\sqrt{2}} e^{-im\theta}$$

$\Longleftrightarrow$

Kinetic k-essence FT

$$\mathcal{L} = m^4 W(X)$$

$$X = g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}$$

Eqs. of motion

$$(\phi^2 g^{\mu\nu}\theta_{,\nu})_{;\mu} = 0$$

$$g^{\mu\nu}\theta_{,\mu}\theta_{,\nu} - \frac{1}{m^2}\frac{dV}{d|\Phi|^2} = 0$$

Eq. of motion

$$(W_X g^{\mu\nu}\theta_{,\nu})_{;\mu} = 0$$

Parametric eq. of state

$$p = m^4 W \qquad \rho = m^4(2XW_X - W)$$

Current conservation

Klein-Gordon current

$$j^{\mu} = i g^{\mu\nu} (\Phi^* \Phi_{,\nu} - \Phi \Phi^*_{,\nu})$$

kinetic k-essence current

$$j^{\mu} = 2 m^2 W_X g^{\mu\nu} \theta_{,\nu}$$

U(1) symmetry ⟺ shift symmetry

$$\Phi \to e^{-i\alpha} \Phi$$

$$\theta \to \theta + c$$

Example: Quartic potential

Scalar field potential

$$V = V_0 \pm m_0^2 \,|\,\Phi\,|^2 + \lambda \,|\,\Phi\,|^4$$

Kinetic k-essence

$$U(Y) = \frac{1}{2}\left(Y \pm \frac{1}{2\lambda}\right)^2 - \frac{1}{8\lambda^2} \quad \Longleftrightarrow \quad W(X) = \frac{1}{2}\left(X \mp \frac{1}{2\lambda}\right)^2$$

Example: Chaplygin gas

Scalar field potential

$$V = m^4 \left(\frac{|\Phi|^2}{m^2} + \frac{m^2}{|\Phi|^2} \right)$$

Scalar Born-Infeld FT

$$U(Y) = \left(Y + \frac{1}{Y} \right) \quad \Longleftrightarrow \quad W(X) = -2\sqrt{1 - X}$$

Age of the Universe

Easy to calculate using the present observed fractions of matter, radiation and vacuum energy.
For a spatially flat Universe from the first Friedmann equation and energy conservation we have

$$H(a) = H_0(\Omega_\Lambda + \Omega_M a^{-3} + \Omega_R a^{-4})^{1/2}$$

$$H_0 = h \times 100\,\mathrm{Gpc/s}^2 = (14.5942\,\mathrm{Gyr})^{-1}, \quad h = 0.67$$

Age of the Universe is then

$$T = \int_0^T dt = \int_0^1 \frac{da}{aH} = \frac{1}{H_0}\int_0^1 \frac{da\sqrt{a}}{(\Omega_\Lambda a^3 + \Omega_M)^{1/2}}$$

$$T \simeq 13.78\,\mathrm{Gyr}$$