Lecture 2 – Braneworld Universe

Strings and Branes
Randal Sundrum Model
Braneworld Cosmology
AdS/CFT and Braneworld Holography

Relativistic Particle Action

PARTICLE is a 0+1-dimdimensional object the dynamics of which in *d*+1-dimensional bulk is described by the relativistic pointlike-particle action

$$S_{\rm part} = -\int \sqrt{ds^2} = -\int d\tau \sqrt{\gamma} = -\int d\tau \sqrt{1 - \dot{x}^2}$$

where

$$ds^2 = G_{ab} dx^a dx^b, \quad \gamma = G_{ab} \frac{\partial x^a}{\partial \tau} \frac{\partial x^b}{\partial \tau}, \qquad a, b = 0, \ldots, d$$

$$\dot{x}^2 = G_{ij} \frac{\partial x^i}{\partial \tau} \frac{\partial x^j}{\partial \tau} \qquad i, j = 1, \ldots, d$$

G_{ab} – metric in the bulk

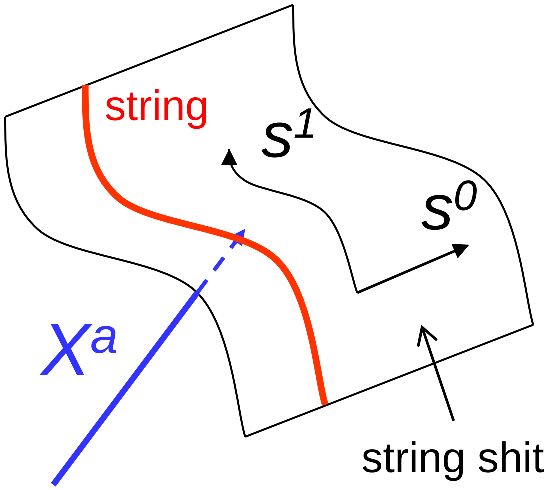
X^a – coordinates in the bulk;

τ – synchronous time coordinate (G_{00}=1)

Strings and (Mem)Branes

STRING is a 1+1-dimdimensional object the dynamics of which in d+1-dimensional bulk is described by the Nambu-Goto action (generalization of the relativistic particle action)

$$S_{\text{string}} = -T\int d\tau d\sigma \sqrt{-\det(g^{\text{ind}}_{\alpha\beta})}$$



where $g_{\alpha\beta}$ is induced metric ("pull back")

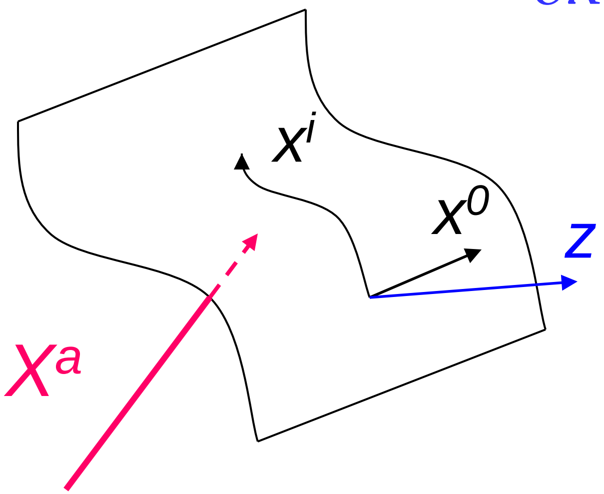
$$g^{\text{ind}}_{\alpha\beta} = G_{ab}\frac{\partial X^a}{\partial s^\alpha}\frac{\partial X^b}{\partial s^\beta} \qquad \alpha,\beta = 0,1$$

G_{ab} – metric in the bulk
X^a – coordinates in the bulk;
$s^0 \equiv \tau$ – timelike coordinate on the string sheet
$s^1 \equiv \sigma$ – spacelike coordinate on the string sheet

p-BRANE is a p+1-dim. object that generalizes the concept of membrane (2-brane) or string (1-brane)

Nambu-Goto action for a 3-brane embedded in a 4+1 dim space-time (bulk)

$$h_{\mu\nu} = G_{ab} \frac{\partial X^a}{\partial x^\mu} \frac{\partial X^b}{\partial x^\nu}$$ – induced metric



G_{ab} – metric in the bulk

X^a – coordinates in the bulk

a,b=0,1,2,3,4

x^μ – coordinates on the brane

μ,ν=0,1,2,3

σ – brane tension

Braneworld universe

Braneworld universe is based on the scenario in which matter is confined on a brane moving in the higher dimensional bulk with only gravity allowed to propagate in the bulk.

N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **429** (1998)
I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **436** (1998)
L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999) 3370 (RS I)
L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999) 4690 (RS II)

We will consider the Randall-Sundrum scenario with a braneworld embedded in a 5-dim asymptotically Anti de Sitter space (AdS_5)

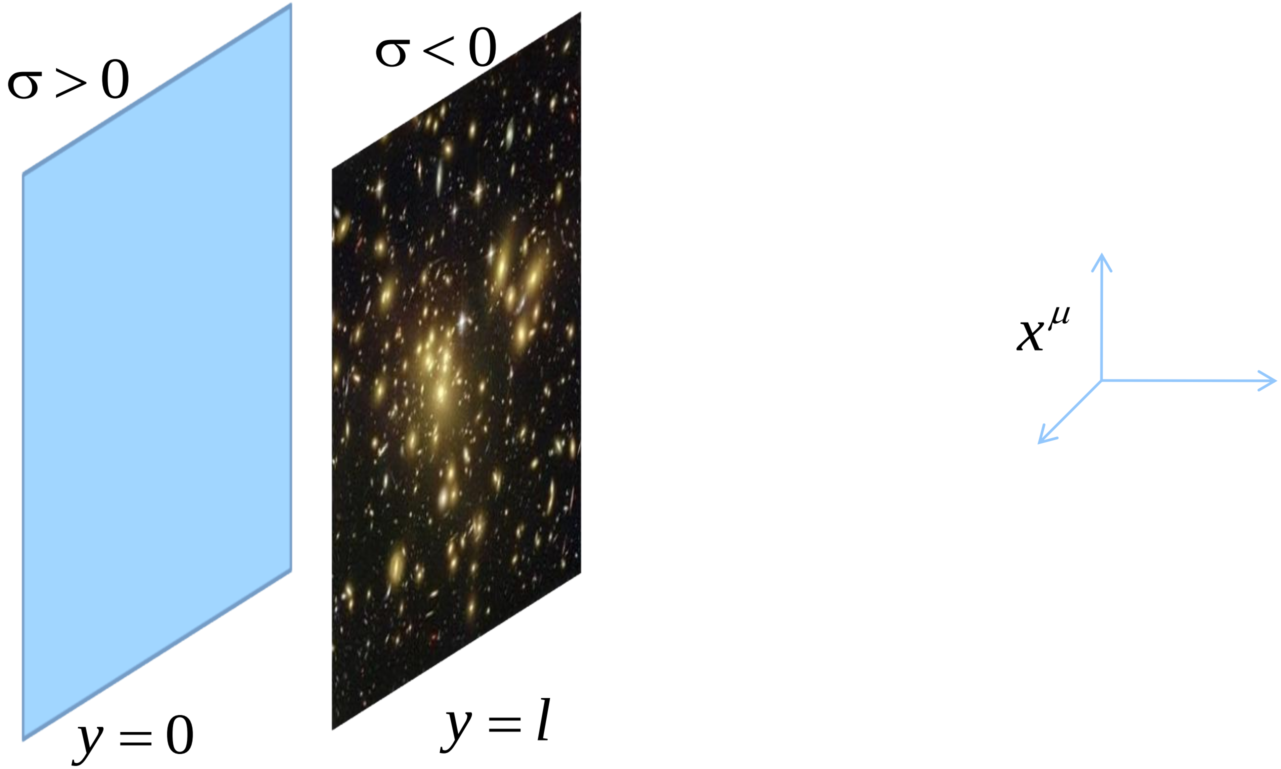
First Randall-Sundrum model (RS I)

RS I was proposed as a solution to the hierarchy problem, in particular between the Planck scale $M_{Pl} \sim 10^{19}$ GeV and the electroweak scale $M_{EW} \sim 10^{3}$ GeV

RS I is a 5-dim. universe with AdS_5 geometry containing two 4-dim. branes with opposite brane tensions separated in the 5th dimension.

The observer is placed on the negative tension brane and the separation is such that the strength of gravity on observer's brane is equal to the observed 4-dim. Newtonian gravity.

Observers reside on the negative tension brane at $y=l$



The coordinate position $y=l$ of the negative tension brane serves as a compactification radius so that the effective compactification scale is $\mu_c = 1/l$

The **conventional** approach to the hierarchy problem is to assume n compact extra dimensions with volume V_n
If their size is large enough compared to the Planck scale, i.e., if

$$\mu_c \sim 1/V_n^{1/n} \ll M_{\mathrm{Pl}}$$

such a scenario may explain the large mass hierarchy between the **electroweak scale** M_{EW} and the **fundamental scale** M of $4+n$ gravity.
In the simplest case, when the $4+n$ dim. spacetime is a product of a 4-dim. spacetime with an n-dim. compact space, one finds

$$M_{\mathrm{Pl}}^2 = M^{2+n} V_n \sim M^{2+n} / \mu_c^n$$

In this way the fundamental $4+n$ scale M could be of the order of M_{EW} if the compactification scale satisfies

$$\mu_c / M_{\mathrm{EW}} \sim (M_{\mathrm{EW}} / M_{\mathrm{Pl}})^{2/n}$$

Unfortunately, this introduces a new hierarchy $\mu_c << M_{\mathrm{EW}}$

Another problem is that there exist a lower limit on the fundamental scale *M* determined by null results in table-top experiments to test for deviations from Newton's law in 4 dimensions, U ~1/ r. These experiments currently probe sub-millimeter scales, so that

$$V^{1/n} \leq 0.1mm \sim \frac{1}{10^{-15}\text{TeV}} \quad \Longrightarrow \quad M \geq \begin{cases} 5\cdot 10^{5}\text{TeV} & \text{for n=1} \\ 3\text{TeV} & \text{for n=2} \end{cases}$$

Long *et al*, Nature **421** (2003).

Stronger bounds for brane-worlds with compact flat extra dimensions can be derived from null results in particle accelerators and in high-energy astrophysics

M. Cavagli`a, "Black Hole and Brane Production in TeV Gravity: A Review"
Int. J. Mod. Phys. A **18** (2003).

S. Hannestad and G.Raffelt, "Stringent Neutron-Star Limits on Large Extra Dimensions"
Phys. Rev. Lett. **88** (2002).

In contrast, RS brane-worlds do not rely on compactification to localize gravity at the brane, but on the curvature of the bulk ("warped compactification"). What prevents gravity from 'leaking' into the extra dimension at low energies is a negative bulk cosmological constant

$$\Lambda_5 = -\frac{6}{\ell^2} = -6k^2$$

ℓ - curvature radius of AdS_5 corresponding to the scale $k = 1/\ell$

Λ_5 acts to "squeeze" the gravitational field closer to the brane. One can see this in Gaussian normal coordinates $X^a = (x^\mu, y)$ on the brane at $y = 0$, for which the AdS_5 metric takes the form

$$ds^2_{(5)} = G_{ab}dX^a dX^b = e^{-2ky}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2$$

Warp factor

The Planck scale is related to the fundamental scale as

$$M_{\text{Pl}} = 2M^3 \int_0^l e^{-2ky} dy = \frac{M^3}{k}\left(1 - e^{-2kl}\right)$$

So that M_{Pl} depends only weakly on l in the limit of large kl. However, any mass parameter m_0 on the observer's brane in the fundamental 5-dim. theory will correspond to the physical mass

$$m = e^{-kl} m_0$$

If kl is of order 10 – 15, this mechanism produces TeV physical mass scales from fundamental mass parameters not far from the Planck scale 10^{19} GeV. In this way we do not require large hierarchies among the fundamental parameters

$$m_0,\ k,\ M,\ \mu_c = 1/l$$

Second Randall-Sundrum model (RS II)

In RSII observers reside on the positive tension brane at $y=0$ and the negative tension brane is pushed off to infinity in the fifth dimension. We shall shortly demonstrate that in this model the Planck mass scale is determined by the curvature of the 5-dimensional space-time $1/k$ and the 5-dim fundamental scale M

$$M_{\mathrm{Pl}}^2 = M^3 2\int_0^\infty e^{-2ky}dy = \frac{M^3}{k}$$

The inverse curvature k serves as the compactification scale and hence the model provides an alternative to compactification.

Second Randall-Sundrum model (RS II)

RS II was proposed as an alternative to compactification of extra dimensions. If extra dimensions were large that would yield unobserved modification of Newton's gravitational law. Eperimental bound on the volume of *n* extra dimensions $V^{1/n} \leq 0.1\,\mathrm{mm}$

Long *et al*, Nature **421** (2003).

RSII brane-world does not rely on compactification to localize gravity at the brane, but on the curvature of the bulk ("warped compactification"). The negative cosmological constant $\Lambda^{(5)}$ acts to "squeeze" the gravitational field closer to the brane. One can see this in Gaussian normal coordinates on the brane at $y = 0$, for which the AdS$_5$ metric takes the form

$$ds^2_{(5)} = G_{ab}\,dX^a dX^b = e^{-2ky}\eta_{\mu\nu}\,dx^\mu dx^\nu - dy^2$$

warp factor

L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999)

“Sidedness”

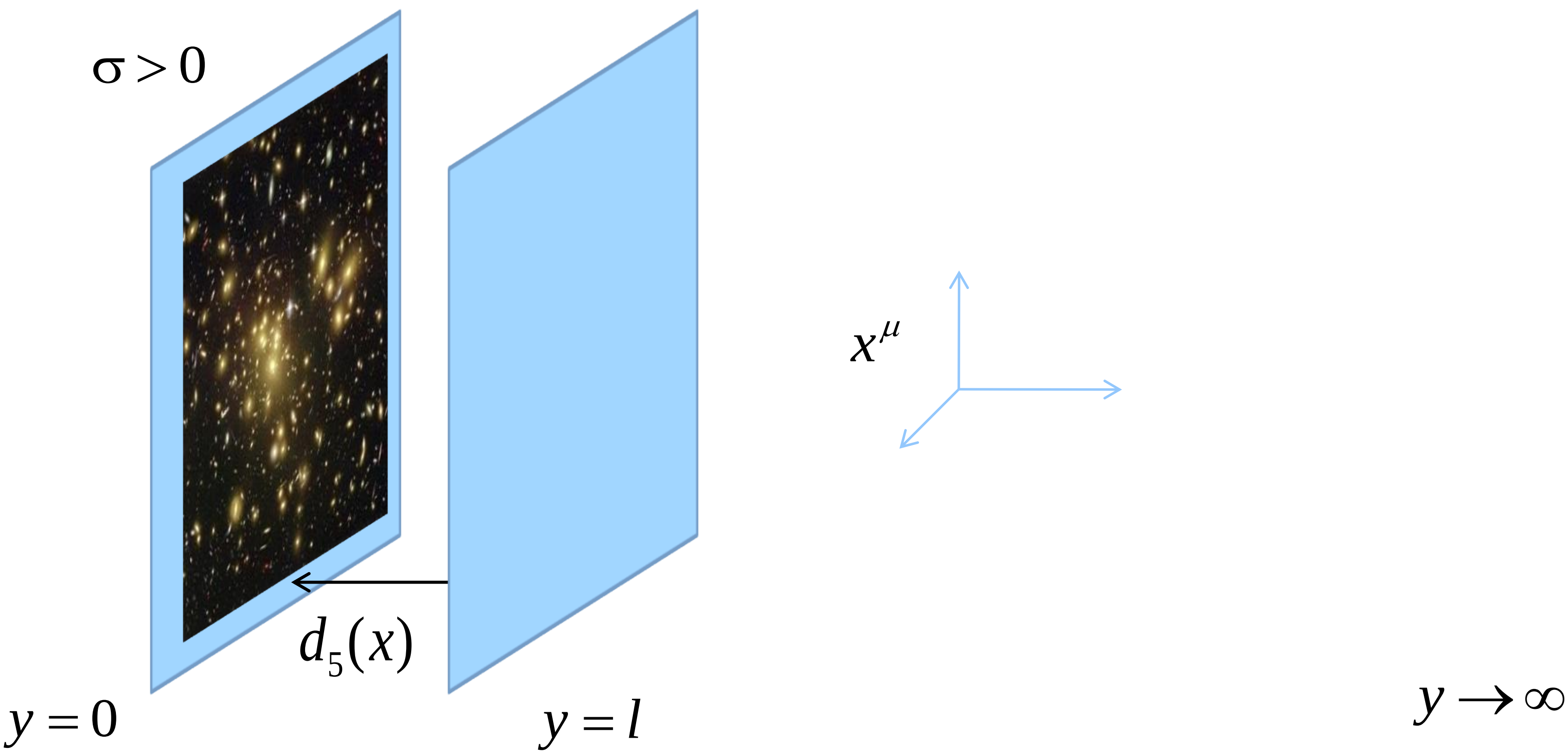
In the original RSII model one assumes the Z_2 symmetry

$$z \leftrightarrow z_{\rm br}/z \qquad \text{or} \qquad y - y_{\rm br} \leftrightarrow y_{\rm br} - y$$

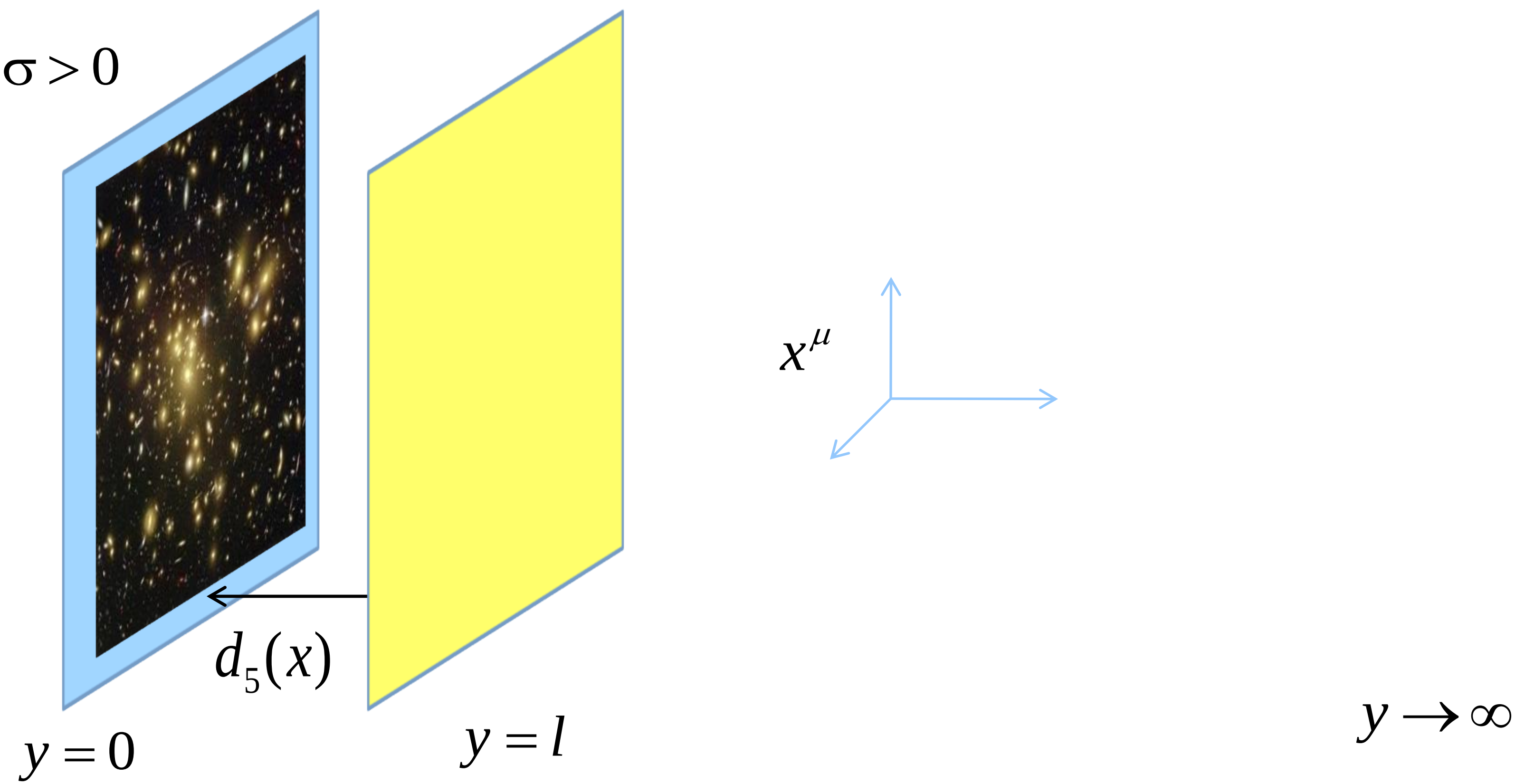
so the region $0 < z \leq z_{\rm br}$ is identied with $z_{\rm br} \leq z < \infty$ with the observer brane at the fixed point $z = z_{\rm br}$. The braneworld is sitting between two patches of AdS_5, one on either side, and is therefore dubbed “two-sided”. In contrast, in the “one-sided” RSII model the region $0 < z \leq z_{\rm br}$ is simply cut off.

As before, the 5-dim bulk is ADS_5 with line element

$$ds^2_{(5)} = G_{ab} dX^a dX^b = e^{-2ky} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2$$



In RSII observers reside on the positive tension brane at *y*=0 and the negative tension brane is pushed off to infinity in the fifth dimension.



AdS bulk is a space-time with negative cosmological constant:

$$\Lambda^{(5)} = -\frac{6}{\ell^2}$$

ℓ - curvature radius of AdS_5

Various coordinate representations:

Fefferman-Graham coordinates

$$ds^2_{(5)} = G_{\mu\nu}dx^\mu dx^\nu = \frac{\ell^2}{z^2}(g_{\mu\nu}dx^\mu dx^\nu - dz^2)$$

$$z = \ell e^{y/\ell}$$

Gaussian normal coordinates

$$ds^2_{(5)} = e^{-2\ell y}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2$$

Schwarzschild coordinates (static, spherically symmetric)

$$ds^2_{(5)} = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2 d\Omega^2_\kappa$$

$$f(r) = \frac{r^2}{\ell^2} + \kappa - \mu \frac{\ell^2}{r^2} \qquad d\Omega^2_\kappa = d\chi^2 + \frac{\sin^2 \sqrt{\kappa}\chi}{\kappa} d\Omega^2$$

$$\kappa = \begin{cases} +1 & \text{closed spherical} \\ 0 & \text{open flat} \\ -1 & \text{open hyperbolic} \end{cases} \qquad \mu = \frac{8G_5 M_{\mathrm{bh}}}{3\pi\ell^2}$$

Relation with *z*

$$\frac{r^2}{\ell^2} = \frac{\ell^2}{z^2} - \frac{\kappa}{2} + \frac{\kappa^2 + 4\mu}{16} \frac{z^2}{\ell^2}$$

Derivation of the RSII model

See Appendix of N.B., Phys. Rev. D 93, 066010 (2016) arXiv:1511.07323

RS model is a 4+1-dim. universe with AdS_5 geometry containing **two** 3-branes with opposite brane tensions separated in the 5th dimension.

$$S = S_{\mathrm{bulk}} + S_{\mathrm{GH}} + S_{\mathrm{br}1} + S_{\mathrm{br}2}$$

The bulk action is given by

$$S_{\mathrm{bulk}} = \frac{1}{8\pi G_5}\int d^5x \sqrt{\det G}\left(-\frac{R^{(5)}}{2} - \Lambda_5\right)$$

where Λ_5 is the negative bulk cosmological constant related to the AdS curvature radius as

The Gibbons-Hawking boundary term is given by an integral over the brane hypersurface Σ

$$S_{\mathrm{GH}} = \frac{1}{8\pi G_5}\int_{\Sigma} d^5x \sqrt{-\det h}\,K$$

where the quantity K is the trace of the extrinsic curvature tensor K_{ab} defined as

$$K_{ab} = h_a^c h_b^d n_{d;c}$$

where n_a is a unit vector normal to the brane pointing towards increasing z, h_{ab} is the induced metric

$$h_{ab} = G_{ab} + n_a n_b \qquad a,b = 0,1,2,3,4$$

and h=det $h_{\mu\nu}$ is its determinant, μ,ν=0,1,2,3

The brane action for each brane is given by the Nambu-Goto action

$$S_{\mathrm{br}} = -\sigma \int d^4x \sqrt{-\det h_{\mu\nu}}$$

Observers reside on the positive tension brane at y=0. The observer total action (including matter) is

$$S_{\mathrm{br}}\,|_{y=0} = -\sigma \int_\Sigma d^4x\sqrt{-h} + \int_\Sigma d^4x\sqrt{-h}\,\mathcal{L}_{\mathrm{matt}}$$

The basic equations are the bulk field equations outside the brane

$$R^{(5)}_{ab} - \frac{1}{2}R^{(5)}G_{ab} = \Lambda_5 G_{ab} \tag{24}$$

and junction conditions

$$[[K^{\mu}_{\nu} - \delta^{\mu}_{\nu}K^{\alpha}_{\alpha}]] = 8\pi G_5(\sigma\delta^{\mu}_{\nu} + T^{\mu}_{\nu}) \tag{25}$$

where the energy momentum tensor T^{μ}_{ν}= diag (ρ, $-p$,$-p$,$-p$) describes matter on the brane and [[f]] denotes the discontinuity of a function f(z) across the brane, i.e.,

$$[[f(z)]] = \lim_{\epsilon\to 0} \left(f(z_{\rm br}+\epsilon) - f(z_{\rm br}-\epsilon)\right)$$

To derive the RSII model solution it is convenient to use Gaussian normal coordinates $x_a = (x_\mu\,, y)$ with the fifth coordinate y related to the Fefferman-Graham coordinate z by $z = \ell e^{y/\ell}$

In the two-sided version with the Z_2 symmetry $y - y_{br} \leftrightarrow y_{br} - y$ one identifies the region $-\infty < y < y_{br}$ with $y_{br} < y < +\infty$. Fwithout loss of generality we may put observer's brane at $y_{br} = 0$.

We start with a simple ansatz for the line element

$$ds^2_{(5)} = \psi^2(y) g_{\mu\nu}(x) dx^\mu dx^\nu - dy^2$$

Where we assume that ψ vanishes at $y=\infty$ and $\psi(0)=1$. Then one finds the relevant components of the Ricci tensor

$$R^{(5)}_{55} = -4\frac{\psi''}{\psi}, \quad R^{(5)}_{5\mu} = 0,$$

$$R^{(5)}_{\mu\nu} = R_{\mu\nu} + \left(3\psi'^2 + \psi\psi''\right) g_{\mu\nu} \tag{26}$$

and the Ricci scallar

$$R^{(5)} = \frac{R}{\psi^2} + 12\frac{\psi'^2}{\psi^2} + 8\frac{\psi''}{\psi}, \tag{27}$$

where the prime $'$ denotes a derivative with respect to y . Using this the action may be brought to the form

$$S[g] = \frac{1}{8\pi G_5} \int d^4x \sqrt{-g} \int dy \left[-\frac{R}{2}\psi^2 - 4(\psi^3\psi')' + 6\psi^2(\psi')^2 - \Lambda^{(5)}\psi^4 \right] + S_{\mathrm{GH}}[g] + S_{\mathrm{br}}[g], \tag{28}$$

The extrinsic curvature is easily calculated using the definition and the unit normal vector $n = (0; 0; 0; 0; 1)$. The nonvanishing components

$$K_{\mu\nu} = n_{\mu;\nu} = -\Gamma^a_{\mu\nu} n_a = \psi\psi' g_{\mu\nu} \qquad (29)$$

The fifth coordinate may be integrated out if $\psi \to 0$ sufficiently fast as we approach $y = \infty$

The functional form of ψ is found by solving the Einstein equations (24) in the bulk. Using the components of the Ricci tensor (26) and Ricci scalar (27) we obtain

$$\frac{R}{2\psi^2} + 6\frac{\psi'^2}{\psi^2} + \Lambda^{(5)} = 0 \qquad (30)$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \left(3\psi'^2 + 3\psi\psi'' + \Lambda^{(5)}\psi^2\right) g_{\mu\nu} \qquad (31)$$

Combining (30) and (31) we find (Exercise No 10)

$$\psi = e^{-y/\ell} \qquad \text{where} \qquad \ell = \sqrt{-6/\Lambda^{(5)}}$$

With this solution, the metric (12) is AdS_5 in normal coordinates Equation (27) reduces to the four-dimensional Einstein equation in empty space

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$$

This equation should follow from the variation of the action (28) with $\mathcal{L}_{\mathrm{matt}} = 0$ after integrating out the fifth coordinate. For this to happen it is necessary that the last three terms in square brackets are canceled by the boundary term and the brane action without matter. Using (29) one finds that the integral of the second term is canceled by the Gibbons-Hawking term. Then, the integration over y fom 0 to ∞ yields

$$S_{\mathrm{b}ulk} + S_{\mathrm{GH}} = \int d^4 x \sqrt{-g} \left\{ -\frac{\gamma \ell}{32\pi G_5} R + \frac{3\gamma}{8\pi \ell G_5} \right\} \qquad (32)$$

For the two branes at y=0 and y=l we find

$$g^{\text{ind}}_{\mu\nu}\,|_{y=0} = g_{\mu\nu} \qquad\qquad g^{\text{ind}}_{\mu\nu}\,|_{y=l} = e^{-2kl} g_{\mu\nu}$$

The total brane contribution is

$$S_{\text{br}}\,|_{y=0} + S_{\text{br}}\,|_{y=l} = -\sigma\int d^4x\sqrt{-g} - \sigma_l\int d^4x\sqrt{-g}\,e^{-4kl}$$

the last term in (32) is canceled by the two brane actions in the limit

$$l \to \infty$$

The cancellations will take place if

$$\sigma = \sigma_0 \equiv \frac{3\gamma}{8\pi G_5 \ell} \qquad (33) \qquad \gamma = \begin{cases} 1, & \text{one-sided RSII,} \\ 2, & \text{two-sided RSII.} \end{cases}$$

This is the RSII fine tuning condition

IIn this way, after integrating out the fifth dimension, the total effective four-dimensional action assumes the form of the standard Einstein-Hilbert action without cosmological constant

$$S = \frac{1}{8\pi G_{\mathrm{N}}} \int d^4x \sqrt{-g} \left(-\frac{R}{2} \right)$$

where G_N is the Newton constant defined by

$$\frac{1}{G_{\mathrm{N}}} = \frac{\gamma}{G_5} \int_0^{\infty} e^{-2y/\ell} dy = \frac{\gamma \ell}{2G_5} \qquad \gamma = \begin{cases} 1 & \text{one-sided} \\ 2 & \text{two-sided} \end{cases}$$

Using this the constant σ_0 in (33) can be expressed as

$$\sigma_0 \equiv \frac{3\gamma}{8\pi G_5 \ell} = \frac{3}{8\pi G_{\mathrm{N}} \ell^2}$$

It may be shown that the fine tuning condition (33) follows directly from the junction conditions (25)

Exercise No 11: Derive the RSII fine tuning condition (33) from the junction conditions

$$[[K^{\mu}_{\nu} - \delta^{\mu}_{\nu} K^{\alpha}_{\alpha}]] = 8\pi G_5 (\sigma \delta^{\mu}_{\nu} + T^{\mu}_{\nu})$$

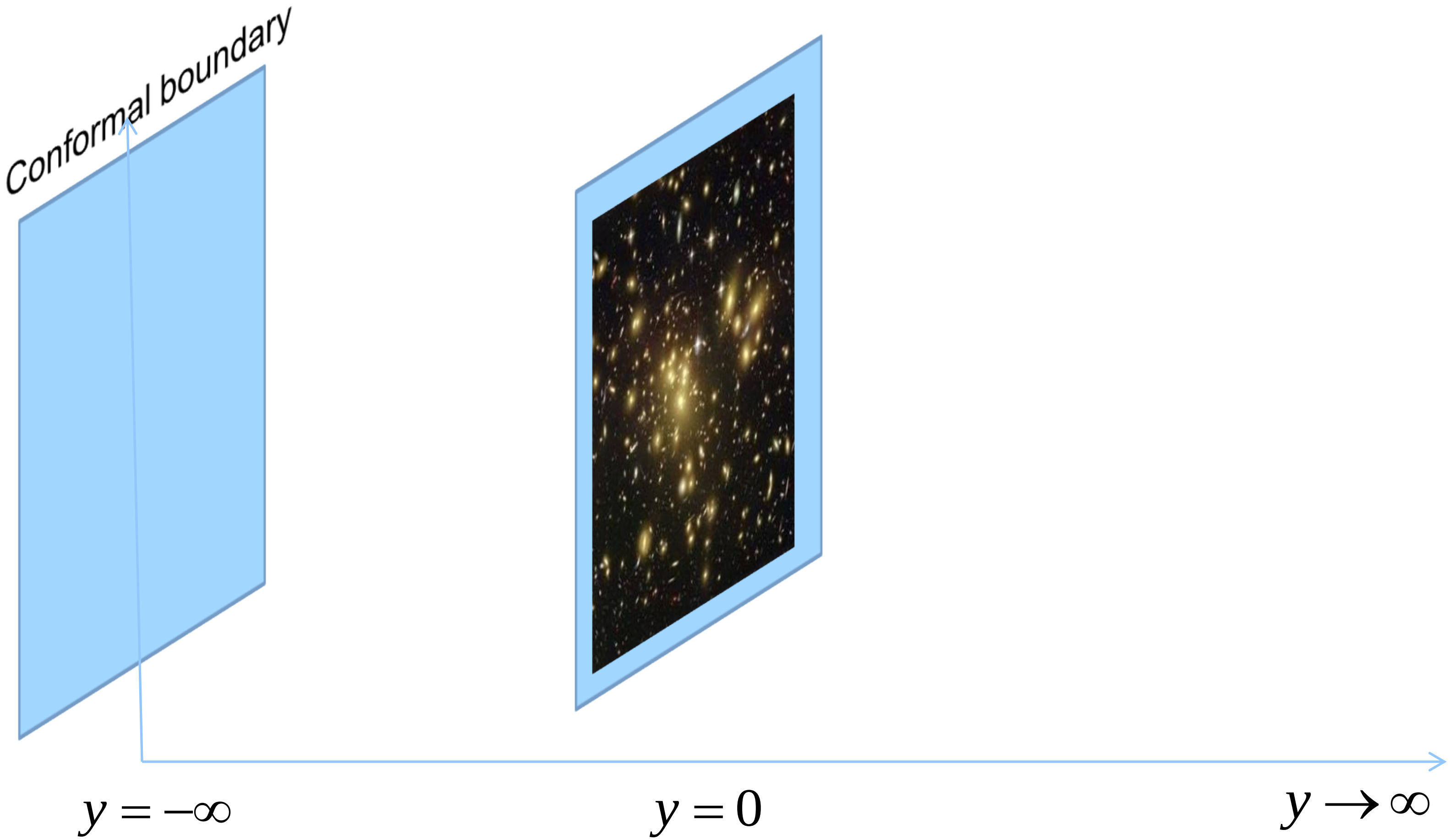
for a brane with no matter at y=0 and the metric

$$ds^2_{(5)} = e^{-2\ell y} \eta_{\mu\nu} dx^{\mu} dx^{\nu} - dy^2$$

N.B., PRD **93**, 066010 (2016) , arXiv:1511.07323

RSII Cosmology – Dynamical Brane

Cosmology on the brane is obtained by allowing the brane to move in the bulk. Equivalently, the brane is kept fixed at y=0 while making the metric in the bulk time dependent.



Simple derivation of the RSII braneworld cosmology

Following J. Soda, Lect. Notes Phys. **828,** 235 (2011) arXiv:1001.1011

See also Appendix in N.B., PRD **93**, 066010 (2016) , arXiv:1511.07323

Consider a time dependent brane hypersurface Σ defined by

$$r - a(t) = 0$$

in AdS-Schwarzschild background. The normal to Σ is

$$n_\mu \propto \partial_\mu (r - a(t)) = (-\partial_t a, 0, 0, 0, 1)$$

Using the normalization $G^{\mu\nu} n_\mu n_\nu = -1$ one finds the nonvanishing components

$$n_t = -\frac{f^{1/2}\partial_t a}{(f^2 - (\partial_t a)^2)^{1/2}} \qquad n_r = \frac{f^{1/2}}{(f^2 - (\partial_t a)^2)^{1/2}}$$

where

$$f(a) = \frac{a^2}{\ell^2} + \kappa - \mu \frac{\ell^2}{a^2} \qquad (34)$$

Then, the induced line element on the brane is

$$ds_{\mathrm{ind}}^2 = n^2(t)dt^2 - a^2(t)d\Omega_k^2$$

where $$n^2 = f(a) - \frac{(\partial_t a)^2}{f(a)}$$

The junction conditions on the brane with matter may be written as

$$K_{\mu\nu}\big|_{r=a-\epsilon} = -\frac{8\pi G_5}{3\gamma}\left[(\sigma+T)g_{\mu\nu} - 3T_{\mu\nu})\right]$$

The $\chi\chi$-component gives

$$\frac{f^{3/2}}{(f^2-(\partial_t a)^2)^{1/2}} = \frac{8\pi G_5}{3\gamma}(\sigma+\rho)a$$

Exercise No 11: Derive the $\chi\chi$ -component of the junction condition

This may be written as

$$\frac{(\partial_t a)^2}{n^2 a^2} + \frac{f}{a^2} = \frac{1}{\ell^2 \sigma_0^2}(\sigma + \rho)^2$$

Hubble expansion rate on the brane

Substituting for *f* the expression (34) we obtain

$$H_{\mathrm{RSII}}^2 + \frac{\kappa}{a^2} = \frac{(\sigma+\rho)^2}{\ell^2 \sigma_0^2} - \frac{1}{\ell^2} + \frac{\mu \ell^2}{a^4}$$

where

$$H_{\mathrm{RSII}}^2 = \frac{(\partial_t a)^2}{n^2 a^2}$$

Employing the RSII fine tuning condition $\sigma = \sigma_0$ we find the effective Friedmann equation

The Friedmann equation on the brane is modified

$$H^2_{\mathrm{RSII}} + \frac{\kappa}{a^2} = \frac{8\pi G_{\mathrm{N}}}{3}\boxed{\rho + \left(\frac{4\pi G_{\mathrm{N}}\ell}{3}\right)^2 \rho^2} + \frac{\mu\ell}{a^4}$$

Quadratic deviation from the standard FRW. Decays rapidly as $\sim a^{-8}$ in the radiation epoch

dark radiation

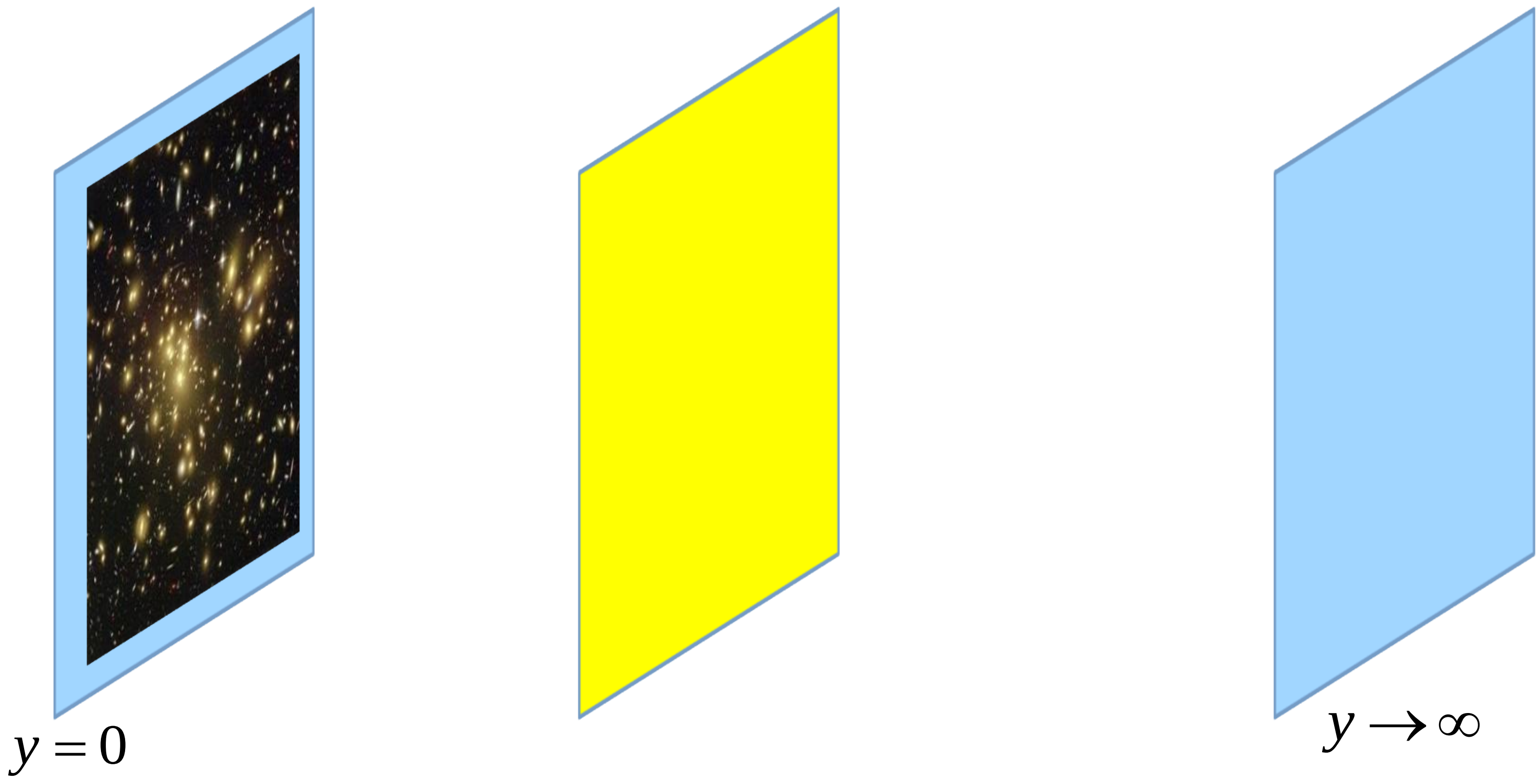
due to a black hole in the bulk – should not exceed 10% of the total radiation content in the epoch of BB nucleosynthesis

RSII cosmology is thus subject to astrophysical tests

Dynamical Brane as a Tachyon

Consider an additional 3-brane moving in the 5-d bulk spacetime with metric

$$ds_{(5)}^2 = \psi^2(y) g^{\mu\nu} dx^{\mu} dx^{\nu} - dy^2$$



The points on the brane are parameterized by $X^M = \left(x^{\mu}, Y\left(x\right)\right)$
The 5-th coordinate *Y* is treated as a dynamical field that depends on *x*.
The brane action is

$$S_{\mathrm{br}} = -\sigma \int d^4 x \sqrt{-\det g_{\mu\nu}^{ind}}$$

Using the induced metric

$$g_{\mu\nu}^{\mathrm{ind}} = g_{(5)MN} X^M_{,\mu} X^N_{,\nu} = \psi^2(Y) g_{\mu\nu} - Y_{,\mu} Y_{,\nu} \qquad (35)$$

one finds

$$S_{\mathrm{br}} = -\sigma \int d^4 x \sqrt{-g}\, \psi^4(Y) \left(1 - \psi^{-2} g^{\mu\nu} Y_{,\mu} Y_{,\nu}\right)^{1/2} \qquad (36)$$

Exercise No 12:

(a)Prove that the following relation holds

$$\det(g_{\mu\nu} - \alpha^2 u_\mu u_\nu) = (1-\alpha^2)\det g_{\mu\nu}$$

for a general metric $g_{\mu\nu}$, unit timelike vector u_μ and $\alpha^2 < 1$

Hint: use a comoving reference frame.

(b) Use (a) to derive the induced metric (36)

Changing *Y* to a new field $\theta = \int dY / \psi(Y)$ we obtain the effective brane action

$$S_{\rm br} = -\int d^4x \sqrt{-g}\, \frac{\sigma}{\chi^4(\theta)} \sqrt{1 - g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}}$$

Where we have defineed

$$\chi(\theta) = \frac{1}{\psi(Y)}$$

This action is of the Born-Infeld type and describes a *tachyon* with potential

$$V(\theta) = \sigma / \chi^4(\theta)$$

Exercise No 13: Show that $V(\theta) \propto \theta^{-4}$ in the AdS_5 background metric

Tachyon as CDM

The effective Born-Infeld Lagrangian

$$\mathcal{L} = -V(\theta)\sqrt{1 - g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}}$$

for the tachyon field θ describes unstable modes in string theory

A. Sen, JHEP **0204** (2002); **0207** (2002).

A typical potential has minima at $\theta = \pm\infty$. Of particular interest is the inverse power law potential $V \propto \theta^{-n}$.

For $n > 2$, as the tachyon rolls near minimum, the pressure $p \equiv \mathcal{L} \to 0$ very quickly and one thus apparently gets pressure-less matter (dust) or cold dark matter.

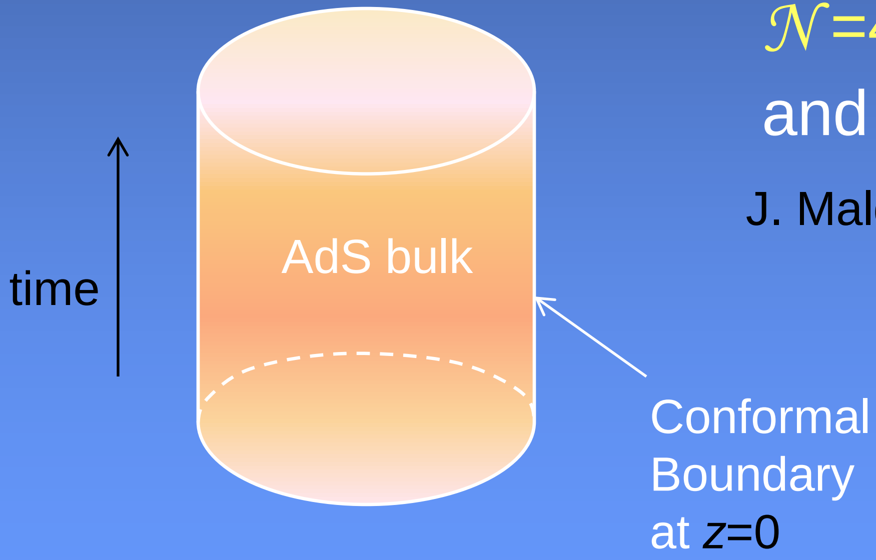
L.R. Abramo and F. Finelli, PLB **575**(2002).

AdS/CFT and Braneworld Holography

AdS/CFT correspondence is a holographic duality between gravity in d+1-dim space-time and quantum CFT on the d-dim boundary. Original formulation stems from string theory:

Equivalence of 3+1-dim
$\mathcal{N}$=4 Supersymmetric YM Theory
and string theory in $AdS_5 \times S_5$

J. Maldacena, Adv. Theor. Math. Phys. **2** (1998)



Examples of CFT:
quantum electrodynamics,
Yang Mills gauge theory,
massless scalar field theory,
massless spin ½ field theory

Why AdS?

Anti de Sitter space is a maximally symmetric solution to Einstein's equations with negative cosmological constant.
In 4+1 dimensions the symmetry group is $AdS_5 \equiv$ **SO(4,2)**

The bulk metric may be represented by (Fefferman-Graham coordinates)

$$ds^2_{(5)} = G_{ab} dX^a dX^b = \frac{\ell^2}{z^2}(g_{\mu\nu} dx^\mu dx^\nu - dz^2)$$

So there is a boundary at z=0. A correspondence between gravity in the bulk and the conformal field theory (CFT) on the boundary of AdS may be expected because the 3+1 boundary conformal field theory is invariant under conformal transformations: Poincare + dilatations + special conformal transformation = conformal group $\equiv$ **SO(4,2)**

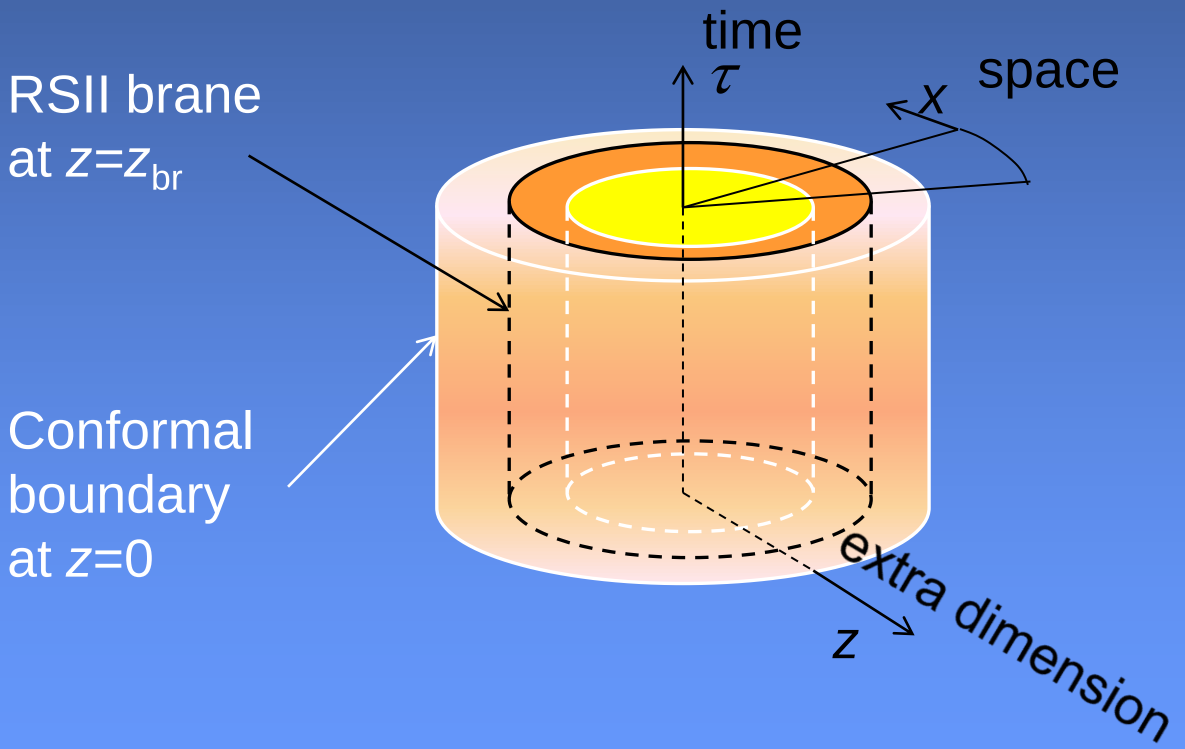
It is sometimes convenient to represented the metric in Gaussian normal coordinates

$$ds^2_{(5)} = e^{-2ky} g_{\mu\nu}(x) dx^\mu dx^\nu - dy^2$$

Warp factor

In the second Randall-Sundrum (RS II) model a 3-brane is located at a finite distance from the boundary of AdS_5.

Foliation of the bulk:



In the original RSII model the region $0 < z \le z_{\mathrm{br}}$ is identied with $z_{\mathrm{br}} \le z < \infty$ with the observer brane at the fixed point $z = z_{\mathrm{br}}$. The braneworld is sitting between two patches of AdS_5, one on either side, and is therefore dubbed "two-sided". In contrast, in the "one-sided" RSII model the region $0 < z \le z_{\mathrm{br}}$ is simply cut off.

1-sided and 2-sided versions are equivalent from the point of view of an observer at the brane. However, in the 1-sided RSII model, by shifting the boundary in the bulk from $z = 0$ to $z = z_{br}$, the model is conjectured to be dual to a cutoff CFT coupled to gravity, with $z = z_{br}$ providing the cutoff. This connection involves a single CFT at the boundary of a single patch of AdS_5. In the 2-sided RSII model one would instead require two copies of the CFT, one for each of the AdS_5 patches.

M. J. Duff and J. T. Liu, Class. Quant. Grav. 18 (2001); Phys. Rev. Lett. 85, (2000)

In the RSII model by introducing the boundary in AdS_5 at $z = z_{br}$ instead of $z = 0$, the model is conjectured to be dual to a cutoff CFT coupled to gravity, with $z = z_{br}$ providing the IR cutoff (corresponding to the UV catoff of the boundary CFT)
The on-shell bulk action is IR divergent because physical distances diverge at $z=0$

$$ds^2_{(5)} = \frac{\ell^2}{z^2}(g_{\mu\nu}dx^\mu dx^\nu - dz^2)$$

A 4-dim asymptotically AdS metric near $z=0$ can be expanded as

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + z^2 g^{(2)}_{\mu\nu} + z^4 g^{(4)}_{\mu\nu} + \cdots$$

Explicit expressions for $g^{(2n)}_{\mu\nu},\ n = 2,4$, in terms of arbitrary $g^{(0)}_{\mu\nu}$

$$g^{(2)}_{\mu\nu} = \frac{1}{2}\left(R_{\mu\nu} - \frac{1}{6} R g^{(0)}_{\mu\nu}\right) \qquad \mathrm{Tr}\, g^{(4)} = -\frac{1}{4}\mathrm{Tr}\,(g^{(2)})^2$$

de Haro, Solodukhin, Skenderis, Comm. Math. Phys. **217** (2001)

We regularize the action by placing the RSII brane near the AdS boundary, i.e., at $z = \varepsilon\ell$, $\varepsilon \ll 1$, so that the induced metric is

$$h_{\mu\nu} = \frac{1}{\varepsilon^2}\left(g^{(0)}_{\mu\nu} + \varepsilon^2 \ell^2 g^{(2)}_{\mu\nu} + \cdots\right)$$

The bulk splits in two regions: $0 \le z \le \varepsilon\ell$, and $\varepsilon\ell \le z \le \infty$. We can either discard the region $0 \le z \le \varepsilon\ell$ (one-sided regularization,) or invoke the Z_2 symmetry and identify two regions (two-sided regularization, $\gamma = 2$).
The regularized bulk action is

$$S^{\mathrm{reg}}_{\mathrm{bulk}} = \gamma S_0 = \frac{\gamma}{8\pi G_5} \int_{z \ge \varepsilon\ell} d^5x \sqrt{\det G}\left(-\frac{R^{(5)}}{2} - \Lambda_5\right) + S_{\mathrm{GH}}$$

We obtain the renormalized boundary action by adding counterterms and taking the limit $\varepsilon \to 0$

$$S_0^{\text{ren}}[g^{(0)}] = \lim_{\varepsilon \to 0} (S_0[G] + S_1[h] + S_2[h] + S_3[h])$$

The necessary counterterms are

$$S_1[h] = -\frac{6}{16\pi G_5 \ell} \int d^4x \sqrt{-h},$$

$$S_2[h] = -\frac{\ell}{16\pi G_5} \int d^4x \sqrt{-h} \left(-\frac{R[h]}{2} \right),$$

$$S_3[h] = -\frac{\ell^3}{16\pi G_5} \int d^4x \sqrt{-h} \frac{\log \epsilon}{4} \left(R^{\mu\nu}[h] R_{\mu\nu}[h] - \frac{1}{3} R^2[h] \right)$$

Hawking, Hertog and Reall, Phys. Rev. D **62** (2000), hep-th/0003052

Now we demand that the variation with respect to $h^{\mu\nu}$ of the total RSII action (the regularized on shell bulk action together with the brane action) vanishes, i.e.,

$$\delta(S_{\text{bulk}}^{\text{reg}}[h] + S_{\text{br}}[h]) = 0$$

Which may be expressed as

matter on the brane

$$\delta\left[\gamma S_0^{\text{ren}} - \gamma S_3 - \left(\sigma - \frac{3\gamma\ell}{8\pi G_5}\right)\int d^4x\sqrt{-h} + \int d^4x\sqrt{-h}\,\mathcal{L}_{\text{matt}} + \frac{\gamma\ell}{16\pi G_5}\int d^4x\sqrt{-h}\,\frac{R[h]}{2}\right] = 0$$

cosmological constant

Einstein Hilbert term

AdS/CFT prescription

$$\delta(S_0^{\text{ren}} - S_3) = \frac{1}{2}\int d^4x\sqrt{-h}\,\langle T_{\mu\nu}^{\text{CFT}}\rangle\,\delta h^{\mu\nu}$$

The variation of the action yields Einstein's equations on the boundary

$$R_{\mu\nu} - \frac{1}{2} R g^{(0)}_{\mu\nu} = 8\pi G_{\mathrm{N}} \left(\gamma \left\langle T^{\mathrm{CFT}}_{\mu\nu} \right\rangle + T^{\mathrm{matt}}_{\mu\nu} \right)$$

This equation (for γ=1) was derived in a different way in

de Haro, Solodukhin, Skenderis, Class. Quant. Grav. **18** (2001) with

$$\langle T^{\mathrm{CFT}}_{\mu\nu} \rangle = -\frac{\ell^3}{4\pi G_5} \left\{ g^{(4)}_{\mu\nu} - \frac{1}{8} \left[(\mathrm{Tr} g^{(2)})^2 - \mathrm{Tr}(g^{(2)})^2 \right] g^{(0)}_{\mu\nu} - \frac{1}{2} (g^{(2)})^2_{\mu\nu} + \frac{1}{4} \mathrm{Tr} g^{(2)} g^{(2)}_{\mu\nu} \right\}$$

de Haro, Solodukhin, Skenderis, Comm. Math. Phys. **217** (2001)

Explicit realization of the AdS/CFT correspondence: the vacuum expectation value of a boundary CFT operator is obtained solely in terms of geometrical quantities of the bulk.

Conformal anomaly

AdS/CFT prescription yields the trace of the boundary stress tensor T^{CFT}

$$\left\langle T^{\mathrm{CFT}\,\mu}{}_{\mu} \right\rangle = \frac{\ell^3}{128\pi G_5}\left(G_{\mathrm{GB}} - C^2\right)$$

$$G_{\mathrm{GB}} = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2$$ Gauss-Bonnet invariant

Weyl tensor squared

compared with the general result from field theory

$$\left\langle T^{\mathrm{CFT}\,\mu}{}_{\mu} \right\rangle = bG_{\mathrm{GB}} - cC^2 + b'\Box R$$

The two results agree if we ignore the last term and identify

$$b = c = \frac{\ell^3}{128\pi G_5}$$

Generally $b \neq c$ because

$$b = \frac{n_{\mathrm{s}} + (11/2)n_{\mathrm{f}} + 62n_{\mathrm{v}}}{360(4\pi)^2} \qquad c = \frac{n_{\mathrm{s}} + 3n_{\mathrm{f}} + 12n_{\mathrm{v}}}{120(4\pi)^2}$$

but in the $\mathcal{N} = 4$ U(N) super YM theory $b=c$ with

$$n_{\mathrm{s}} = 6N^2, n_{\mathrm{f}} = 4N^2, n_{\mathrm{v}} = N^2$$

The conformal anomaly is correctly reproduced if we identify

$$\frac{\ell^3}{G_5} = \frac{2N^2}{\pi}$$