Hamiltonian Method in the Braneworld

Neven Bilić
Ruđer Bošković Institute
Zagreb

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Lecture 1 - Preliminaries

Legendre Transformation

Consider an arbitrary smooth function $f(x)$. We can define another function $g(u)$ such that

$$f(x) + g(u) = xu \qquad (1)$$

where the variables x and y (called conjugate variables) are related via

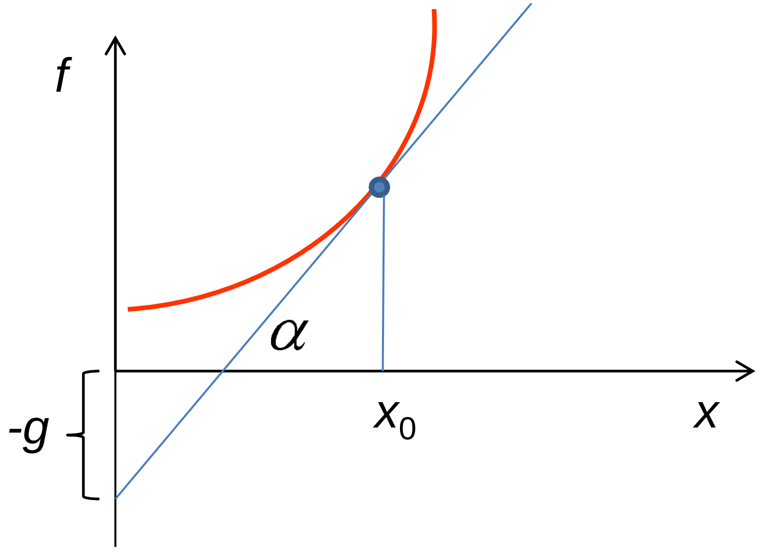
$$y = \frac{\partial f}{\partial x}, \qquad x = \frac{\partial g}{\partial u}$$

Proof:

At an arbitrary point x_0 the function $f(x)$ can be locally represented by

$$f(x_0) = u_0 x_0 - g \qquad \text{where} \qquad u_0 = \left.\frac{\partial f}{\partial x}\right|_{x_0}$$

Simple geometric meaning



$$u_0 = \tan\alpha$$

g depends on u_0

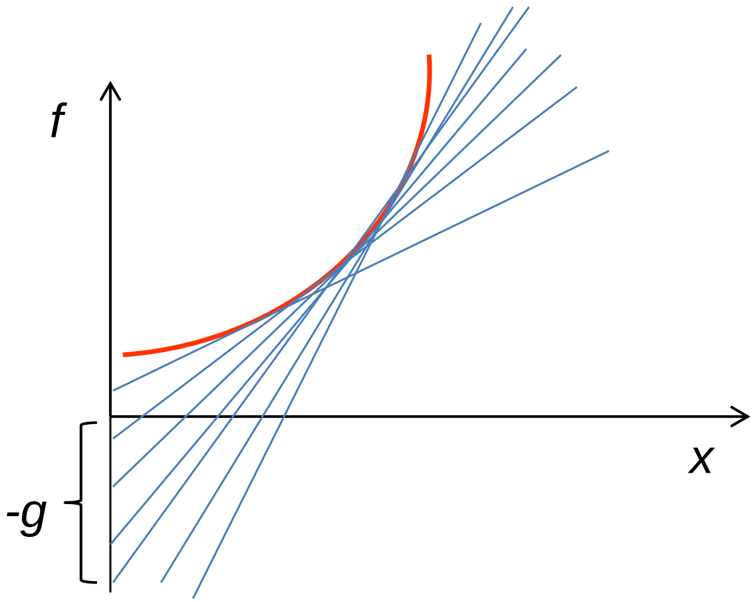
By the symmetry of (1) we can also write

$$g(u_0) = x_0 u_0 - f$$

where

$$x_0 = \left.\frac{\partial g}{\partial u}\right|_{u_0}$$

Exercise No 1: Prove this using the geometry in the figure



By variing x_0 the function f(x) may be regarded as the envelope of tangents

g is an implicit function of u

The generalization to n dimensions is straightforward:

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} u_i x_i - g(u_1, u_2, \ldots, u_n)$$

with

$$u_i = \frac{\partial f}{\partial x_i}, \qquad x_i = \frac{\partial g}{\partial u_i}$$

Applications

1) Classical mechanics

$$\mathcal{H}(p_i) = \sum_j q_j p_j - \mathcal{L}(q_i) \qquad p_i = \frac{\partial \mathcal{L}}{\partial q_i}, \qquad q_i = \frac{\partial \mathcal{H}}{\partial p_i}$$

2) Thermodynamics

$$F(T,V) = E(S,V) - TS \qquad S = -\frac{\partial F}{\partial T}, \qquad T = \frac{\partial E}{\partial S}$$

F – Helmholz free energy, *E* – internal energy, *S* – entropy

1) Quantum firld theory

$$\Gamma[\phi] = W[J] - \int dx J(x)\phi(x) \qquad J(x) = -\frac{\delta \Gamma}{\delta \phi(x)}, \qquad \phi(x) = \frac{\delta W}{\delta J(x)}$$

W – generating functional of connected Green's functions

Γ – generating functional of 1-particle irreducible Green's functions

Basic Thermodynamics

The conjugate variables:

V – volume $\quad$ p – pressure

S – entropy $\quad$ T – temperature

N – particle number $\quad$ μ – chemical potential

We will distinguish the grandcanonical from canonical ensemble

Canonical: particle number N fixed. Related potentials are the Gibbs free energy G and internal energy E

Grandcanonical: chemical potential μ fixed. N is not conserved. Related potentials are the grandcanonical thermodynamic potential $\Omega=-pV$ and internal energy E

Canonical ensemble

The **Gibbs free energy** *G* is related to the internal energy E via Legendre transformation

$$G(T,p) = E(S,V) - TS + pV \qquad (2)$$

together with the conditions

$$S = -\frac{\partial G}{\partial T}, \qquad V = \frac{\partial G}{\partial p}, \qquad T = \frac{\partial E}{\partial S}, \qquad p = -\frac{\partial E}{\partial V} \qquad (3)$$

The well known thermodynamic identities follow:

$$dE = TdS - pdV \qquad (4)$$

TDS equation

$$dG = -SdT + Vdp \qquad (5)$$

Exercise No 2: Derive (4) and (5) from (2) and (3)

Canonical ensemble

The potentials *G* and *E* are related the **Helmholz free energy** *F* via respective Legendre transformation,

$$F(T,V) = G(T,p) - pV \qquad F(T,V) = E(S,V) - TS$$

So that $$p = \frac{\partial F}{\partial V} \qquad S = -\frac{\partial F}{\partial T}$$

Grandcanonical ensemble

The grandcanonical **thermodynamic potential** $\Omega=-pV$ is related to the internal energy $E=\rho V$, entropy $S=sV$ and particle number $N=nV$ via Legendre transformation

$$\Omega(T,\mu) = E(S,N) - TS - \mu N$$

which may be expressed locally (dividing by V)

$$p(T,\mu) = Ts + \mu n - \rho(s,n) \qquad (6)$$

together with the conditions

$$s = \frac{\partial p}{\partial T}, \qquad n = \frac{\partial p}{\partial \mu}, \qquad T = \frac{\partial \rho}{\partial s}, \qquad \mu = \frac{\partial \rho}{\partial n} \qquad (7)$$

Introducing the specific **enthalpy** (or the specific **heat content**)

$$w = \frac{p+\rho}{n} \qquad (8)$$

Two useful thermodynamic identities follow

$$dw = Td\frac{s}{n} + \frac{1}{n}dp \qquad (9)$$

TDS equation

$$d\frac{p}{T} = nd\frac{\mu}{T} - \rho d\frac{1}{T} \qquad (10)$$

Gibbs-Duhem relation

Exercise No 3: Derive (9) and (10) from (6)-(8)

Basic Fluid Mechanics

We will assume that matter is a **perfect fluid** described by the energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}$$

u_μ – fluid velocity

$T_{\mu\nu}$ – energy-momentum tensor

p – pressure

ρ – energy density

The energy momentum conservation

$$T^{\mu\nu}{}_{;\nu} = 0$$

yields, as its longitudinal part $u_\mu T^{\mu\nu}{}_{;\nu} = 0$, the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0, \qquad (11)$$

and, as its transverse part, the Euler equation

$$(p+\rho)\dot{u}_\mu - p_{,\mu} + \dot{p}u_\mu = 0 \qquad (12)$$

where

$$3H = u^\nu{}_{;\nu}, \qquad \dot{\rho} = u^\nu \rho_{,\nu}, \quad \dot{p} = u^\nu p_{,\nu}, \quad \dot{u}^\mu = u^\nu u^\mu{}_{;\nu},$$

H – expansion (Hubble expansion rate in cosmology)

Exercise No 4: Derive (11) and (12)

Velocity field

It is convenient to parameterize the four-velocity u_μ in terms of three-velocity components. To do this, we use the projection operator

$$g_{\mu\nu} - t_\mu t_\nu$$

where t_μ is the time translation unit vector $t_\mu = g_{\mu 0} / \sqrt{g_{00}}$. We split up the vector u_μ in two parts: one parallel with and the other orthogonal to t_μ :

$$u_\mu = \gamma t_\mu + (g_{\mu\nu} - t_\mu t_\nu) u^\nu, \qquad \gamma = t^\mu u_\mu$$

yielding

$$u^\mu = \gamma \left(\frac{1}{\sqrt{g_{00}}} - \frac{g_{0j} v^j}{g_{00}}; v^i \right), \qquad (13)$$

$$u_\mu = \gamma \left(\sqrt{g_{00}}; \frac{g_{0i}}{\sqrt{g_{00}}} - v_i \right),$$

where

$$v_i = \gamma_{ij} v^j, \quad v^2 = v^i v_i, \quad \gamma^2 = (1 - v^2)^{-1} \quad \gamma_{ij} = \frac{g_{0i} g_{0j}}{g_{00}} - g_{ij}$$

Exercise No 5: Derive (13)

Isentropic and adiabatic fluid

A flow is said to be ***isentropic*** when the specific entropy *s/n* is constant, i.e., when

$$(s/n)_{,\mu} = 0 \tag{14}$$

and is said to be ***adiabatic*** when *s/n* is constant along the flow lines, i.e., when

$$u^\mu (s/n)_{,\mu} = 0$$

As a consequence of (14) and the thermodynamic identity (8) (TdS equation) the Euler equation (12) simplifies to

$$u^\mu \left[(w u_\mu)_{;\nu} - (w u_\nu)_{;\mu} \right] = 0 \tag{15}$$

In this case, we may introduce a scalar function φ such that

$$w u_\mu = \varphi_{,\mu} \tag{16}$$

which obviously solves (15)

Exercise No 6: Derive (15) from (12) and (14)

NB1: The solution (16) $wu_{\mu} = \varphi_{,\mu}$ is the relativistic analogue of potential flow in nonrelativistic fluid dynamics

L.D. Landau, E.M. Lifshitz, Fluid Mechanics, Pergamon, Oxford,1993.

NB2: The potential flow (16) implies the isentropic Euler equation (15) but not the other way round. However if the fluid is ***isentropic*** and ***irrotational***, then equations (15) and (16) are equivalent. The fluid is said to be ***irrotational*** if its vorticity vanishes. The vorticity is defined as

$$\omega_{\mu\nu} = h^{\rho}_{\mu} h^{\sigma}_{\nu} u_{[\rho;\sigma]} \quad \text{where} \quad h^{\mu}_{\nu} = \delta^{\mu}_{\nu} - u^{\mu} u_{\nu}$$

Vanishing vorticity, i.e., $\omega_{\mu\nu}=0$, implies

$$(wu_{\mu})_{;\nu} - (wu_{\nu})_{;\mu} = 0$$

A.H. Taub, Relativistic Fluid Mechanics, Ann. Rev. Fluid Mech. 10 (1978) 301

This equation is satisfied if and only if $wu_{\mu} = \varphi_{,\mu}$ and the Euler equation (15) is satisfied *identically*

Lagrangian and Hamiltonian

The dream of all physicists is a comprehensive fundamental theory, which is often in popular scientific literature called the "**theory of everything**". Of course, nobody expects that this theory provides answers to all the issues, for example, the cause of a cancer, how the mind works, and so on. From the **theory of everything** we only require to explain basic processes in nature. Today most physicists share the following view of the world: the laws of nature are unambiguously described by the principle of some unique **action** (or Lagrangian) that fully defines the vacuum, the spectrum of elementary particles, forces and symmetries.

Classical description

Consider a single selfinteracting scalar field θ with with a general action

$$S = \int d^4x \sqrt{-\det g}\,\mathcal{L}(X,\theta) \quad \text{where } X = g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}$$

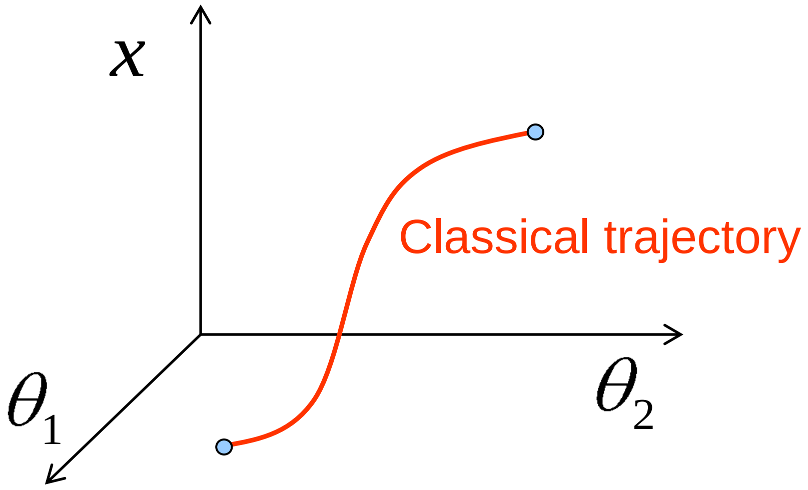
One can easily generalize this to more fields $\theta_1, \theta_2, \cdots$ with

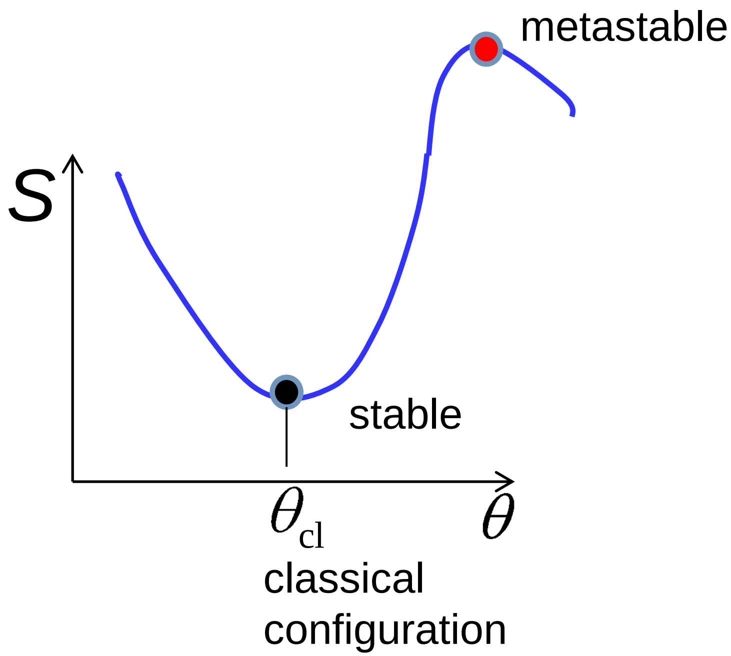
$$X_i = g^{\mu\nu}\theta_{i,\mu}\theta_{i,\nu}$$

The principle of least action

$$\delta S = 0$$

yields classical equations of motion

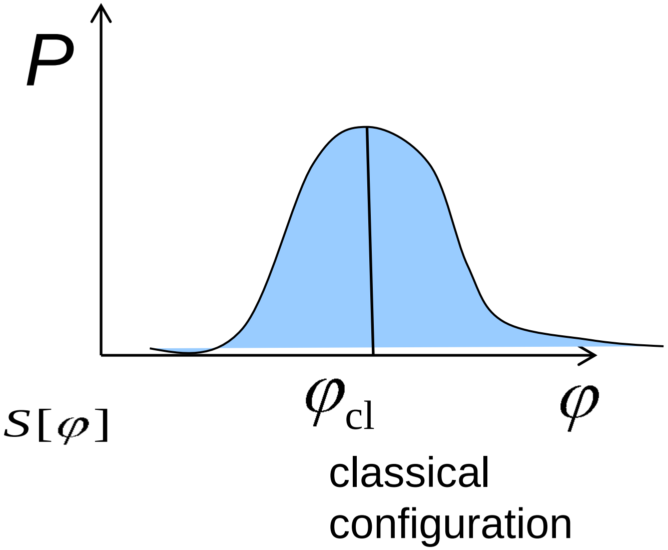




Quantum description

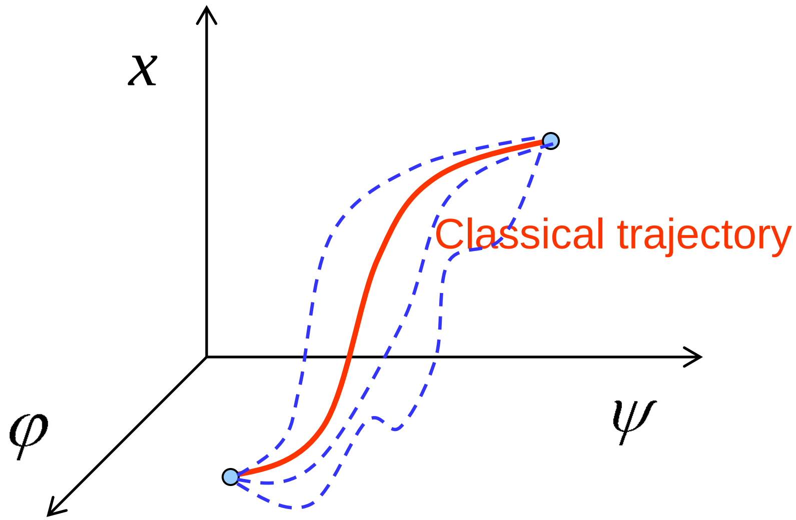
Through the action *S* we define a probability distribution

$$P[\varphi_i] = \frac{e^{-S[\varphi_i]}}{Z}$$



Z = Partition function – path integral

$$Z = \sum_i e^{-S[\varphi_i]} \Longrightarrow \int [d\varphi] e^{-S[\varphi]}$$



From the requirement δS=0 one finds the **classical** equations of motion

$$\left(\frac{\partial \mathcal{L}}{\partial \theta_{i,\mu}}\right)_{;\mu} = \frac{\partial \mathcal{L}}{\partial \theta_i} \qquad (17)$$

To each scalar field θ_i one can associate a current

$$J_{(i)\mu} = \mathcal{L}_{X_i} \theta_{i,\mu}$$ where

If $\mathcal{L}$ does not depend on θ_i, the associated current is conserved, i.e.

$$J^{\mu}_{(i);\mu} = 0$$

This current conservation law follows immediately from (17) and is related to the shift symmetry

$$\theta_i \rightarrow \theta_i + c$$

Hamiltonian

The canonical **Hamiltonian** is defined through the usual Legendre transformation that involves the conjugate variables $\theta_{,0}$ and π^0

$$\mathcal{H}_{\mathrm{can}}(\pi^0, \theta_{,i}, \theta) = \pi^0 \theta_{,0} - \mathcal{L}(\theta_{,0}, \theta_{,i}, \theta), \quad i = 1, 2, 3$$

where

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \theta_{,0}} \qquad \theta_{,0} = \frac{\partial \mathcal{H}_{\mathrm{can}}}{\partial \pi^0}$$

We shall use a covariant definition (de Donder-Weyl Hamiltonian)

$$\mathcal{H}(\pi^\mu, \theta) = \pi^\nu \theta_{,\nu} - \mathcal{L}(\theta_{,\mu}, \theta) \tag{18}$$

where

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \theta_{,\mu}} \qquad \theta_{,\mu} = \frac{\partial \mathcal{H}}{\partial \pi^\mu} \tag{19}$$

It may be easily shown that the covariant definition of $\mathcal{H}$ coincides with the energy density ρ in the field theoretical description of a perfect relativistic fluid which we will discuss shortly

Historically, the covariant Hamiltonian was first introduced by De Donder 1930 and Weyl 1935 in the so called polysymplectic formalizm

Th. De Donder, *Th´eorie Invariantive Du Calcul des Variations,* Gaultier-Villars & Cia., Paris, France (1930).

H. Weyl, Annals of Mathematics 36, 607 (1935)

Recent references

J. Struckmeier, A. Redelbach, *Covariant Hamiltonian Field Theory*, Int. J. Mod. Phys. E17: 435-491, 2008, arXiv:0811.0508

C. Cremaschini and M.Tessarotto, ``Manifest Covariant Hamiltonian Theory of General Relativity,"' Appl. Phys. Res. 8, 60 (2016), arXiv:1609.04422

Examples:

1) The standard scalar field Lagrangian

$$\mathcal{L} = \frac{1}{2}X - V(\theta) \qquad \mathcal{H} = \frac{1}{2}X + V(\theta)$$

2) The Born-Infeld (tachyon) Lagrangian

$$\mathcal{L} = -U(\theta)\sqrt{1-X} \qquad \mathcal{H} = \frac{U(\theta)}{\sqrt{1-X}}$$

Field theoretical description of a fluid

Using a general single field Lagrangian

$$S = \int d^4x \sqrt{-\det g}\, \mathcal{L}(X,\theta)$$ where $$X = g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}$$

we define the energy momentum tensor

$$T_{\mu\nu} = \frac{2}{\sqrt{-\det g}} \frac{\delta S}{\delta g^{\mu\nu}} = 2\mathcal{L}_X \theta_{,\mu}\theta_{,\nu} - \mathcal{L} g_{\mu\nu}$$ where $$\mathcal{L}_X \equiv \frac{\partial \mathcal{L}}{\partial X}$$

This may be written in a perfect fluid form provided $X>0$

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu - p g_{\mu\nu}$$

where we identify the velocity, pressure and energy density

$$u_\mu = \frac{\theta_{,\mu}}{\sqrt{X}} \qquad p = \mathcal{L} \qquad \rho = 2X\mathcal{L}_X - \mathcal{L}$$

Exercise No 7: Prove $\rho = \mathcal{H}$

Hamilton field equations

Suppose for definitness that $\mathcal{L}$ depends on two fields Θ and Φ and and their respective kinetic terms $X = g^{\mu\nu}\Theta_{,\mu}\Theta_{,\nu}$ and $Y = g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu}$
We introduce the canonical conjugate momentum fields

$$\pi^{\mu}_{\Phi} = \frac{\partial \mathcal{L}}{\partial \Phi_{,\mu}} = 2\mathcal{L}_Y g^{\mu\nu}\Phi_{,\nu} \qquad \pi^{\mu}_{\Theta} = \frac{\partial \mathcal{L}}{\partial \Theta_{,\mu}} = 2\mathcal{L}_Y g^{\mu\nu}\Theta_{,\nu}$$

For timelike $\Phi_{,\nu}$ and $\Theta_{,\nu}$ we may also define the norms

$$\pi_{\Phi} = \sqrt{g_{\mu\nu}\pi^{\mu}_{\Phi}\pi^{\nu}_{\Phi}} \qquad \pi_{\Theta} = \sqrt{g_{\mu\nu}\pi^{\mu}_{\Theta}\pi^{\nu}_{\Theta}} \qquad (20)$$

We shall assume that the dependence on kinetic terms can be separated, i.e., that the Lagrangian can be expressed as

$$\mathcal{L}(X,Y,\Theta,\Phi) = \mathcal{L}_1(X,\Theta,\Phi) + \mathcal{L}_2(Y,\Theta,\Phi)$$

The corresponding energy momentum tensor

$$T_{\mu\nu} = 2\frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} - \mathcal{L} g_{\mu\nu} = 2\mathcal{L}_X^{(1)}\pi_{\Theta\mu}\pi_{\Theta\nu} - g_{\mu\nu}\mathcal{L}^{(1)} + 2\mathcal{L}_Y^{(2)}\pi_{\Phi\mu}\pi_{\Phi\nu} - g_{\mu\nu}\mathcal{L}^{(2)}$$

may be expressed as a sum of two perfect fluids

$$T_{\mu\nu} = (p_1 + \rho_1)u_{1\mu}u_{1\nu} + (p_2 + \rho_2)u_{2\mu}u_{2\nu} - (p_1 + p_2)g_{\mu\nu}$$

with $u_{1\mu} = \pi_{\Theta\mu}/\pi_\Theta \qquad u_{2\mu} = \pi_{\Phi\mu}/\pi_\Phi$

$$p_1 = \mathcal{L}_X^{(1)} \qquad \rho_1 = 2X\mathcal{L}_X^{(1)} - \mathcal{L}^{(1)}$$

$$p_2 = \mathcal{L}_Y^{(2)} \qquad \rho_2 = 2Y\mathcal{L}_Y^{(2)} - \mathcal{L}^{(2)}$$

$\mathcal{H}$ is related to $\mathcal{L}$ through the Legendre transformation

$$\mathcal{H}(\Theta, \Phi, \pi_\Phi^\mu, \pi_\Theta^\mu) = \pi_\Phi^\nu \Phi_{,\nu} + \pi_\Theta^\nu \Theta_{,\nu} - \mathcal{L}(\Theta, \Phi, \Phi_{,\mu}, \Theta_{,\mu}) \quad (21)$$

It may be easily verified that the Hamiltonian equals to the total energy density

$$\mathcal{H} = T^{\mu}{}_{\mu} + 3\mathcal{L} = \rho_1 + \rho_2$$

The field variables are constrained by

$$\Phi_{,\mu} = \frac{\partial \mathcal{H}}{\partial \pi_\Phi^\mu} \qquad \Theta_{,\mu} = \frac{\partial \mathcal{H}}{\partial \pi_\Theta^\mu}$$

$$\pi_\Phi^\mu = \frac{\partial \mathcal{L}}{\partial \Phi_{,\mu}} \qquad \pi_\Theta^\mu = \frac{\partial \mathcal{L}}{\partial \Theta_{,\mu}} \quad (22)$$

Now we multiply the first and second equation by u_1^μ and u_2^μ, respectively, take a covariant divergence of the next two equations, and use the obvious relations

$$\frac{\partial \mathcal{H}}{\partial \Theta} = -\frac{\partial \mathcal{L}}{\partial \Theta} \qquad \frac{\partial \mathcal{H}}{\partial \Phi} = -\frac{\partial \mathcal{L}}{\partial \Phi}$$

We obtain a set of four 1st order Hamilton's diff. equations

$$\dot{\Theta} = \frac{\partial \mathcal{H}}{\partial \pi_\Theta} \qquad \dot{\pi}_\Theta + 3H_1\pi_\Theta = -\frac{\partial \mathcal{H}}{\partial \Theta}$$
$$\dot{\Phi} = \frac{\partial \mathcal{H}}{\partial \pi_\Phi} \qquad \dot{\pi}_\Phi + 3H_1\pi_\Phi = -\frac{\partial \mathcal{H}}{\partial \Phi} \tag{23}$$

where $\dot{\Phi} \equiv u_1^\mu \Phi_{,\mu}, \;\; \dot{\Theta} \equiv u_2^\mu \theta_{,\mu}, \;\; \dot{\pi}_\Phi = u_1^\mu \pi_{\Phi,\mu}, \;\; \dot{\pi}_\Theta = u_1^\mu \pi_{\Theta,\mu}$

and $3H_1 = u^\mu_{1\,;\mu} \qquad 3H_2 = u^\mu_{2\,;\mu}$ are fluid expansions

Exercise No 8: Derive eqs. (23) from (21) and (22)

Basic Cosmology

- General relativity – gravity

$$G = T$$

Matter *determines the space-time geometry*

Geometry *determines the motion of matter*

- Homogeneity and isotropy of space – approximate property on very large scales (~Glyrs today)
- Fluctuations of matter and geometry in the early Universe cause structure formation (stars, galaxies, clusters ...)

General Relativity – Gravity

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} + g_{\mu\nu}\Lambda$$

$g_{\mu\nu}$ - metric tensor

$R_{\mu\nu}$ - Riemann curvature tensor

Λ - cosmological constant

$T_{\mu\nu}$ - energy – momentum tensor

Cosmological principle

Homogeneity and isotropy of space

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$$

$a(t)$ – cosmological scale

the curvature constant *k* takes on the values 1, 0, or -1, for a closed, flat, or open universe, respectively.

Friedman equations

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho$$

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

$$H \equiv \frac{\dot{a}}{a}$$ expansion rate

k > 0 closed
$= 0$ flat
< 0 open

$$T^{\mu\nu}{}_{;\nu} = 0 \quad \Rightarrow \quad \rho \propto a^{-3(1+w)} \qquad w = p/\rho$$

Various kinds of cosmic fluids with different w

- radiation $p_R = \rho_R/3$ $w = 1/3$ $\rho \propto a^{-4}$
- dust $p_M = 0$ $w = 0$ $\rho \propto a^{-3}$
- vacuum $p_\Lambda = -\rho_\Lambda$ $w = -1$ $\rho \propto a^{0}$

Density of Matter in Space

The best agreement with cosmologic observations are obtained by the models with a flat space

According to Einstein's theory, a flat space universe requires critical matter density ρ_{cr} today

$\rho_{cr} \approx 10^{-29}$ g/cm^3

$\Omega = \rho / \rho_{cr}$ ratio of the actual to the critical density

For a flat space Ω=1

What does the Universe consist of?

From astronomical observations:

luminous matter (stars, galaxies, gas ...) $\rho_{lum}/\rho_{cr} \leq$ **0.5%**

From the light element abundances and comparison with the Big Bang nucleosynthesis:

baryonic matter(protons, neutrons, nuclei) $\rho_{Bar}/\rho_{cr} \leq$ **5%**

Total matter density fraction $\Omega_M = \rho_M/\rho_{cr} \approx$ **0.31**

Accelerated expansion and comparison of the standard Big Bang model with observations requires that the **dark energy** density (vacuum energy) today $\Omega_\Lambda = \rho_\Lambda / \rho_{cr}$ **= 0. 69%**

Density fractions of various kinds of matter today with respect to the total density

$$\Omega_B = \frac{\rho_B}{\rho_{\text{tot}}} \approx 0.05 \quad \Omega_{\text{DM}} = \frac{\rho_{\text{DM}}}{\rho_{\text{tot}}} \approx 0.26 \quad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{\text{tot}}} \approx 0.69$$

These fractions change with time but for a spatially flat Universe the following always holds:

$$\rho_{\text{tot}} = \rho_{\text{crit}}$$

Age of the Universe

Easy to calculate using the present observed fractions of matter, radiation and vacuum energy.
For a spatially flat Universe from the first Friedmann equation and energy conservation we have

$$H(a) = H_0(\Omega_\Lambda + \Omega_M a^{-3} + \Omega_R a^{-4})^{1/2}$$

$$H_0 = h \times 100\,\mathrm{Gpc/s}^2 = (14.5942\,\mathrm{Gyr})^{-1}, \quad h = 0.67$$

The age of the Universe *T* can be calculated using

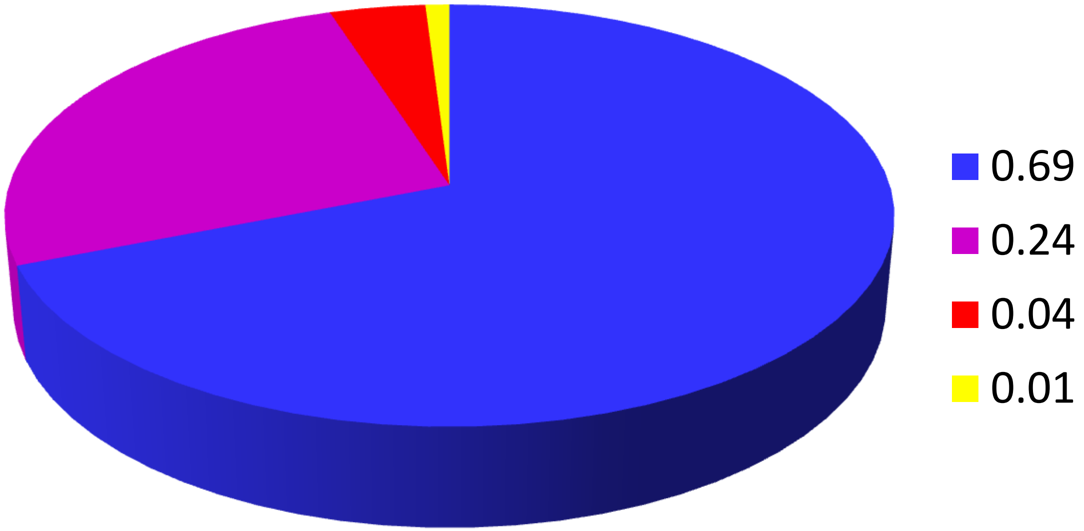
$$T = \int_0^T dt = \int_0^1 \frac{da}{aH}$$

Exercise No 9: Calculate *T* using $\Omega_\Lambda=0.69$, $\Omega_M=0.31$, $\Omega_R=0$.

Dark nature of the Universe – Dark Matter

According to present recent observations (Planck Satellite Mission):

More than 99% of matter is not luminous



- Out of that less then 5% is ordinary ("baryonic")
- About 24% is Dark Matter
- About 69% is Dark Energy (Vacuum Energy)

Hot DM refers to low-mass neutral particles that are still relativistic when galaxy-size masses ($\sim 10^{12} M_{\odot}$) are first encompassed within the horizon. Hence, fluctuations on galaxy scales are wiped out. Standard examples of hot DM are neutrinos and majorons. They are still in thermal equilibrium after the **QCD deconfinement** transition, which took place at $T_{\mathrm{QCD}} \approx 150$ MeV. Hot DM particles have a cosmological number density comparable with that of microwave background photons, which implies an upper bound to their mass of a few tens of eV.

Warm DM particles are just becoming nonrelativistic when galaxy-size masses enter the horizon. Warm DM particles interact much more weakly than neutrinos. They decouple (i.e., their mean free path first exceeds the horizon size) at $T \gg T_{QCD}$. As a consequence, their mass is expected to be roughly an order of magnitude larger, than hot DM particles. Examples of warm DM are keV sterile neutrino, axino, or gravitino in soft supersymmetry breaking scenarios.

Cold DM particles are already nonrelativistic when even globular cluster masses ($\sim 10^6 M_\odot$) enter the horizon. Hence, their free path is of no cosmological importance. In other words, all cosmologically relevant fluctuations survive in a universe dominated by cold DM. The two main particle candidates for cold dark matter are the lowest supersymmetric *weakly interacting massive particles* (WIMPs) and the *axion*.

Dark Energy

Because gravity acts as an attractive force between astrophysical objects we expect that the expansion of the Universe will slowly decelerate.

However, recent observations indicate that the Universe expansion began to accelerate since about 5 billion years ago.

Repulsive gravity?

Accelerated expansion $\Rightarrow \Lambda \neq 0$

One possible explanation is the existence of a fluid with negative pressure such that

$$p + 3\rho < 0$$

and in the second Friedmann equation the universe acceleration $\ddot{a}$ becomes positive

cosmological constant Λ = vacuum energy density with equation of state *p=-ρ.* Its negative pressure may be responsible for accelerated expansion!

New term: Dark Energy – fluid with negative pressure - generalization of the concept of vacuum energy

Problems with Λ

1) Fine tuning problem. The calculation of the vacuum energy density in field theory of the Standard Model of particle physics gives the value about 10^{120} times higher than the value of Λ obtained from observations. One possible way out is fine tuning: a rather unnatural assumption that all interactions of the standard model of particle physics somehow conspire to yield cancellation between various large contributions to the vacuum energy resulting in a small value of Λ , in agreement with observations

2) Coincidence problem. Why is this *fine tuned* value of Λ such that DM and DE are comparable today, leaving one to rely on anthropic arguments?

Time dependence of the DE density

Another important property of DE is that its density does not vary with time or **very weakly** depends on time. In contrast , the density of ordinary matter varies rapidly because of a rapid volume expansion.

The rough picture is that in the early Universe when the density of matter exceeded the density of DE the Universe expansion was slowing down. In the course of evolution the matter density decreases and when the DE density began to dominate, the Universe began to accelerate.

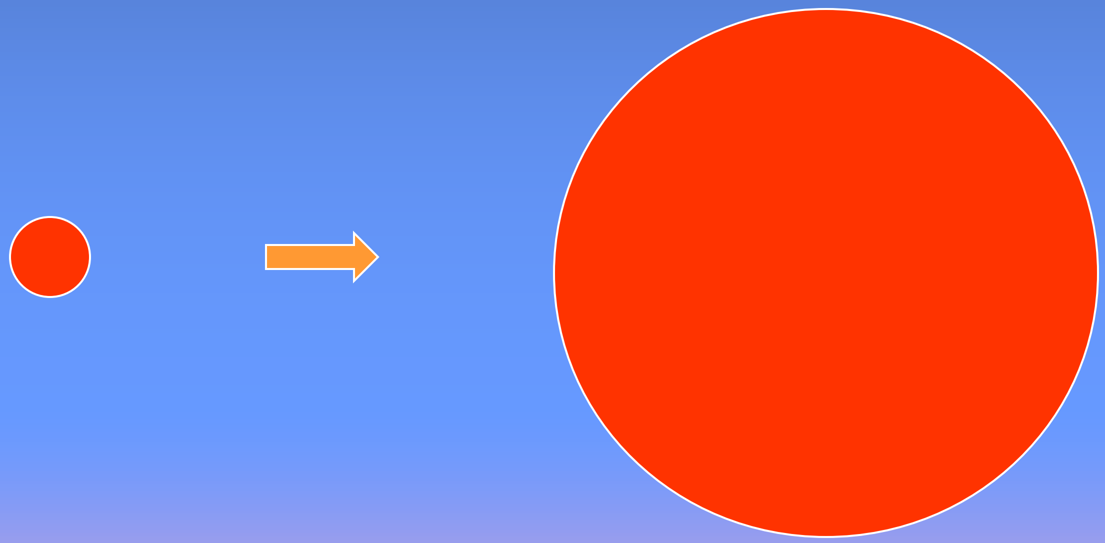
Most popular models of dark energy

- Cosmological constant – vacuum energy density. Energy density does not vary with time.
- Quintessence – a scalar field with a canonical kinetic term. Energy density varies with time.
- Phantom quintessence – a scalar field with a negative kinetic term. Energy density varies with time.
- k-essence – a scalar field whose Lagrangian is a general function of kinetic energy. Energy density varies with time.
- Quartessence – a model of unifying of DE and DM. Special subclass of k-essence. One of the popular models is the so-called *Chaplygin gas*

Early Universe - Inflation

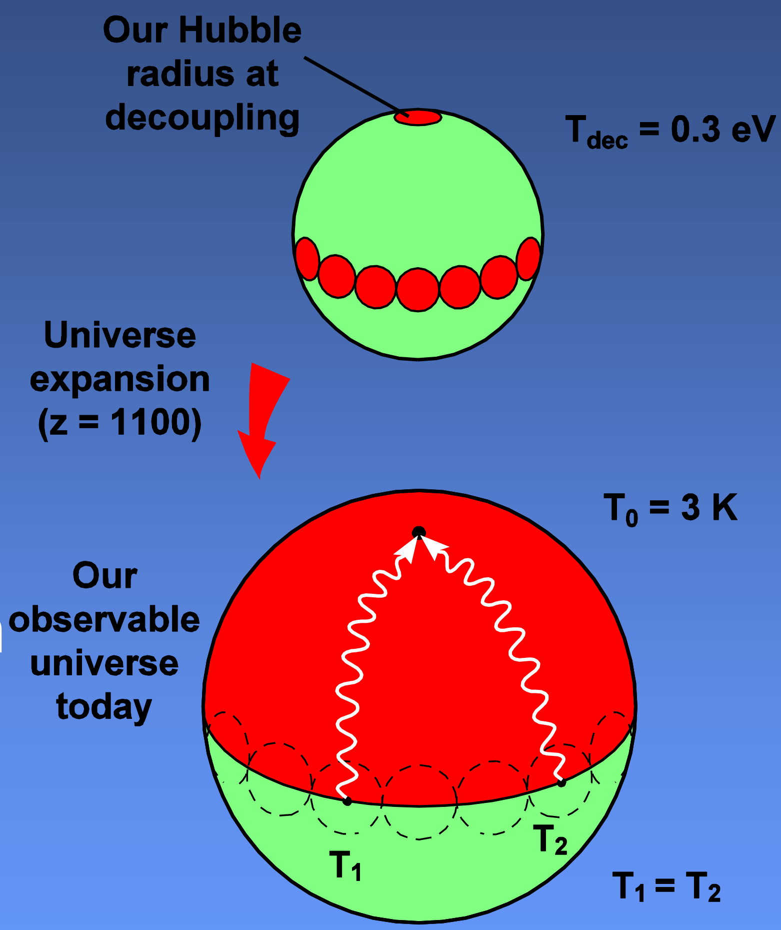
A short period of **inflation follows** – very rapid expansion -

10^{25} times in 10^{-32} s.



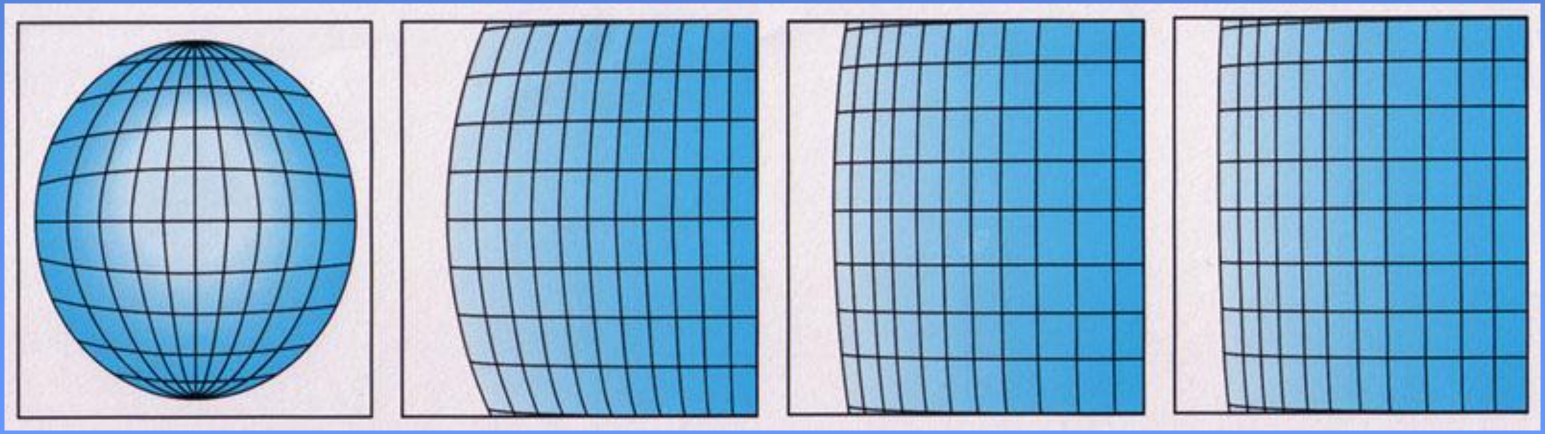
The horizon problem.

Observations of the cosmic microwave background radiation (CMB) show that the Universe is homogeneous and isotropic. The problem arises because the information about CMB radiation arrive from distant regions of the Universe which were not in a causal contact at the moment when radiation had been emitted – in contradiction with the observational fact that the measured temperature of radiation is equal (up to the deviations of at most about 10^{-5}) in all directions of observation.



The flatness problem

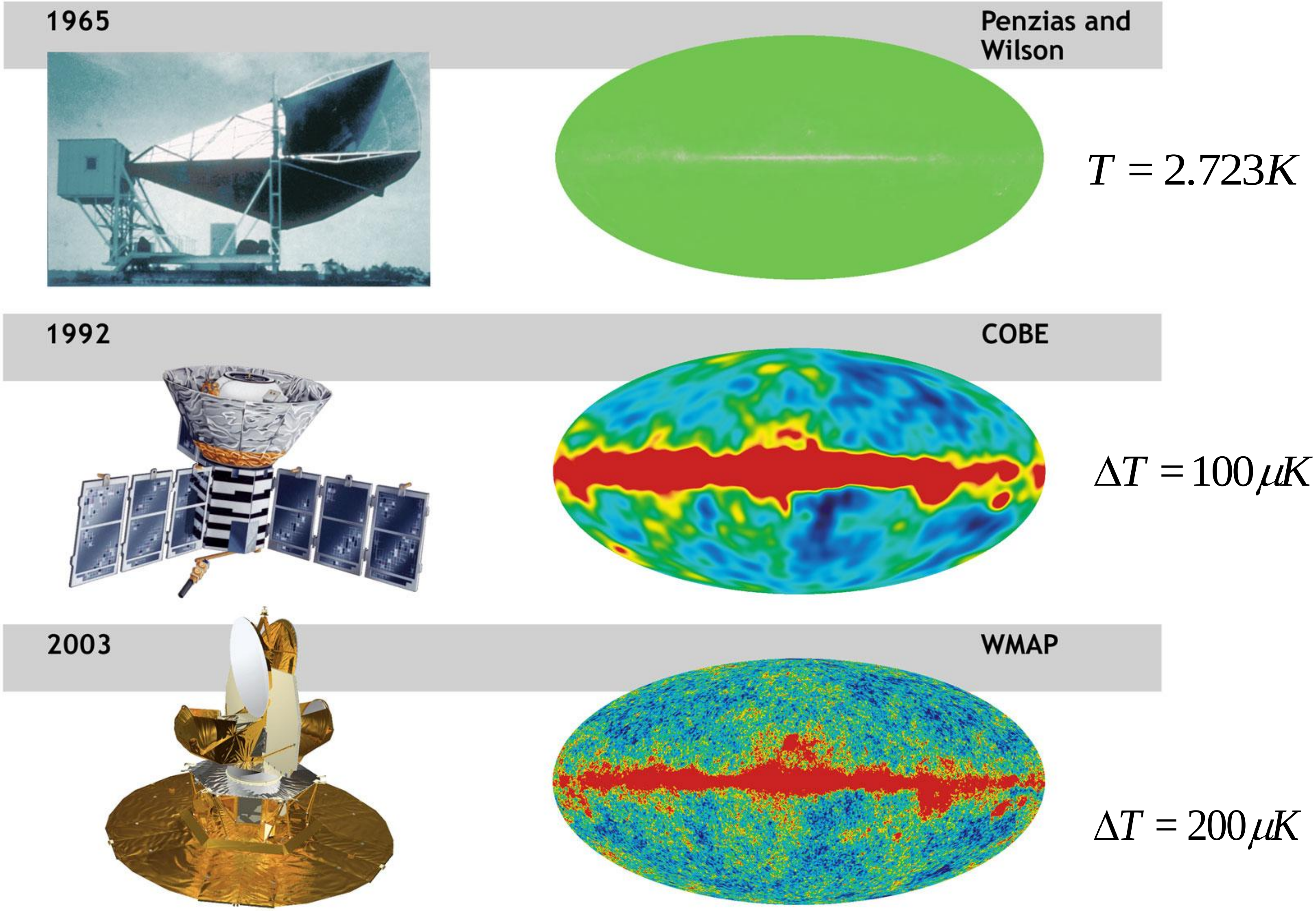
Observations of the average matter density, expansion rate and fluctuations of the CMB radiation show that the Universe is flat or with a very small curvature today. In order to achieve this, a “fine-tuning“ of the initial conditions is needed, which is rather unnatural. The answer is given by inflation:



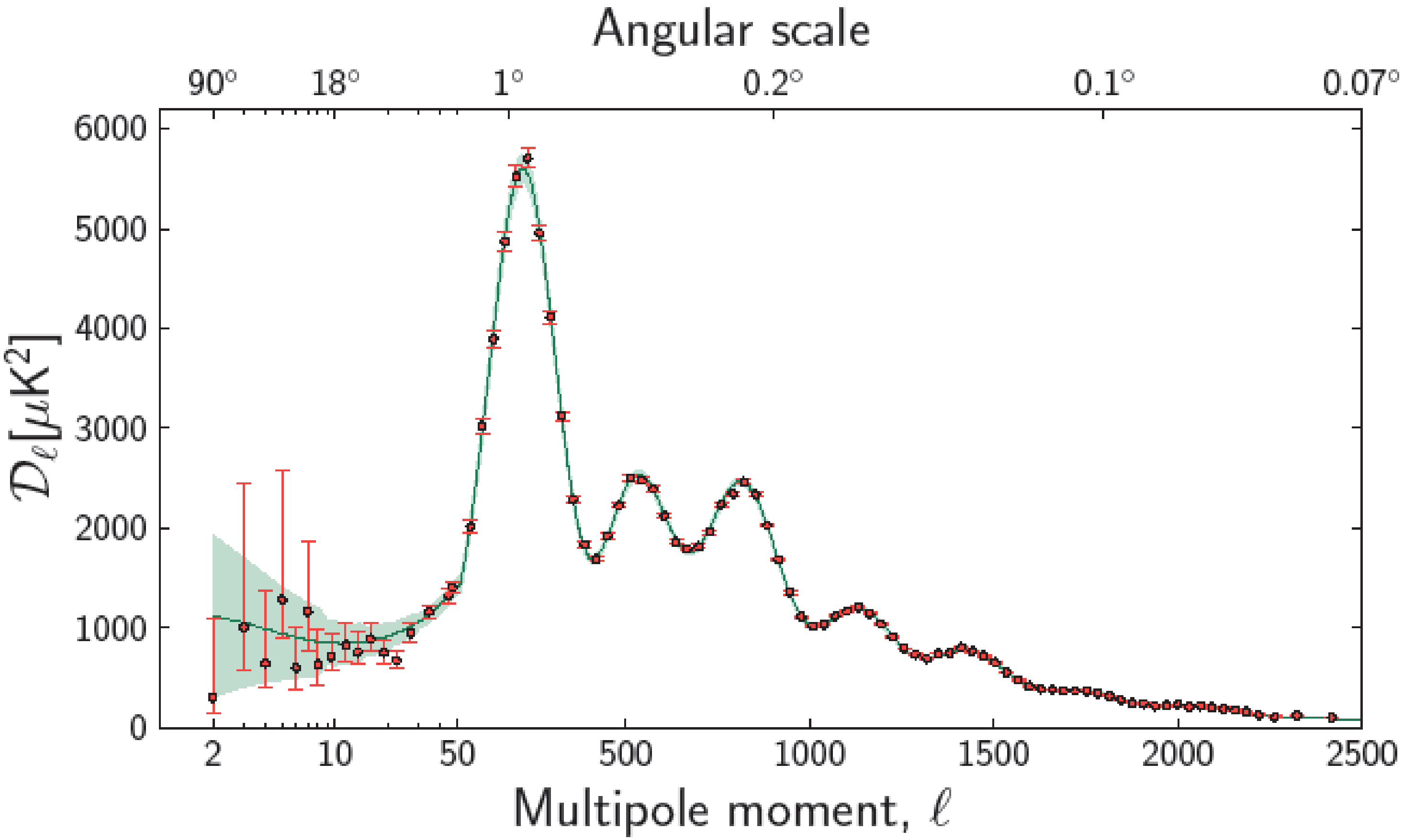
10^{-5} $v = Hd,$

The initial density perturbations

The question is how the initial deviations from homogeneity of the density are formed having in mind that they should be about 10^{-5} in order to yield today’s structures (stars, galaxies, clusters). The answer is given by inflation: perturbations of density are created as quantum fluctuations of the inflaton field.



Measuring CMB; the temperature map of the sky.



Angular (multipole) spectrum of the fluctuations of the CMB
(Planck 2013)